The Split Delivery Vehicle Routing Problem with Small Capacity

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Abstract

In this paper we consider a vehicle routing problem where a fleet of homogeneous vehicles must serve a set of customers, the demand of which can take any integer number possibly greater than the vehicle capacity. Each time a vehicle visits a customer it collects an integer quantity. The objective is to minimize the total distance travelled by the vehicles to serve all the customers. This problem is known in the literature as the *Split Delivery Vehicle Routing Problem* (SDVRP). In this paper we show that, if some specific conditions on the distances are satisfied, the problem with a vehicle capacity equal to 2 is solvable in polynomial time. When the distances are symmetrical and satisfy the triangle inequality, this problem is reducible, by making direct trips to the depot, to a problem where each customer demand is strictly lower than 2. On the other side we show that the problem with vehicle capacity $k \geq 3$ is NP-hard. When the capacity is equal to 3 the problem is reducible only when the distances satisfy the sharpened triangle inequality with $\alpha = 2/3$.

Keywords: Split Delivery Vehicle Routing; triangle inequality; computational complexity.

Introduction

In this paper we consider a vehicle routing problem where a fleet of homogeneous vehicles has to serve a set of customers. The demands of the customers can take any integer value possibly greater than the capacity of the vehicles which is represented by k, $k \in Z^+$. A customer may need to be served more than once (multiple customer visits), contrary to what is usually assumed in vehicle routing problems. Each time a vehicle visits a customer it collects an integer quantity. No constraint on the number of available vehicles is considered. Each vehicle starts from and returns to the depot at the end of each tour. The objective is to minimize the total distance travelled by the vehicles to serve all the customers. We will

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present the problem as a collection problem but, for reasons of symmetry, it can also be seen as a distribution problem.

This problem is known in the literature as the *Split Delivery Vehicle Routing Problem* (SDVRP) which is a VRP where the constraint for which each customer must be visited only once is relaxed. In this paper we consider the particular case where the demand of each customer, as well as the quantity collected by the vehicles when visiting a customer, is an integer number. Note that when each customer has unitary demand a particular case of the *Capacitated Vehicle Routing Problem* is obtained (for a survey on vehicle routing problems see (Christofides, Mingozzi and Toth, 1979) and (Golden and Assad, 1988)). When the demand of each customer is lower than or equal to the capacity of the vehicles, the difference between the SDVRP and the classical CVRP is that in the former problem a customer is allowed to be visited more than once.

In the following, we will refer to the SDVRP with vehicles of capacity k, $k \in Z^+$, general distances and demand n_i for each customer i (n_i can be greater than k), as the k-SDVRP.

In (Christofides, 1985) it is shown that if the sum of the three smallest demands of the customers exceeds the capacity of the vehicles then the CVRP can be solved in polynomial time: actually, as each vehicle can serve no more than 2 customers, the problem is reducible to a *matching problem,* for which there exists a polynomial time algorithm. This result implies that the CVRP with unitary demands and capacity of the vehicles equal to 2 can be solved in polynomial time. To the best of our knowledge, the particular case of the CVRP in which vehicles have all a capacity $k = 3$ has not been studied yet. A case with limitation on the discrete capacity of the vehicles (measured in skips) has been studied in (De Meulemeester, Laporte, Louveaux and Semet, 1997), where the authors have considered a pick-up and delivery problem where each vehicle can transport only one skip and they have solved it on a large-scale real instance. In (Ball, Bodin, Baldacci and Mingozzi, 2000) the authors have studied a similar problem, referred to as the *Rollon-Rolloff Vehicle Routing Problem* (RRVRP). In (Altinkemer and Gavish, 1990) and (Altinkemer and Gavish, 1987) Altinkemer and Gavish have presented heuristics with constant error guarantees for the CVRP with equal (all customers have a demand equal to 1) or unequal (customers can have different demands) weights. In (Altinkemer and Gavish, 1991) a parallel savings heuristic for the VRP is proposed and a special case where each vehicle has a discrete capacity of 2 and the customers have all demand equal to one is considered. The SDVRP has been studied in (Dror and Trudeau, 1989) and (Dror and Trudeau, 1990) where Dror and Trudeau have analyzed the savings generated by allowing split deliveries in a vehicle routing problem. They have also shown that when the distances satisfy the triangle inequality then there exists an optimal solution for the SDVRP where no pairs of tours have 2 vertices in common. In (Dror, Laporte and Trudeau, 1994) the authors have found a set of valid inequalities for the SDVRP. Real applications of this problem are studied in (Mullaseril, Dror and Leung, 1997) and (Sierksma and Tijssen, 1998) where heuristic algorithms are presented. In (Belenguer, Martinez and Mota, 2000) a lower bound is proposed for the SDVRP where the demand of each customer is less than the capacity of the vehicles and the quantity collected by the vehicles when visiting a customer is an integer number. Frizzell and Giffin (Frizzell and Giffin, 1995) have considered a particular case of the SDVRP with grid network distances and time windows for which they present a mathematical formulation and a heuristic algorithm.

In the following sections we will define the demand of a customer as the number of skips to be collected, so that each vehicle can transport 2 or 3 skips. The motivation for this problem comes from a real-case application we have studied, in which vehicles had to collect full skips from customers and each vehicle had a capacity of 2 (or 3) skips.

The paper is organized as follows. In Section 1 we introduce the general problem and define the property of *problem reducibility* as well as the main notation and definitions used in the paper. The problem reduction is obtained when an optimal solution exists in which, for each customer, as many as possible full load direct trips from the depot are made. This is shown, in Sections 2 and 3, for $k = 2$ and $k = 3$ respectively, under specific conditions on the distances and implies that the real routing problem to be solved is a problem where the demand of each customer is lower than the capacity of the vehicles. While for $k = 2$ in the reduced case a customer can be visited only once, this may not be the case for $k = 3$. More precisely, in Section 2 we show that, when the distances are symmetrical and satisfy the triangle inequality, the 2-SDVRP can be reduced to the case where each customer has a demand of 1, maintaining, or possibly reducing, the number of customers. In Section 3 we analyze the 3-SDVRP. We prove that, under more restrictive conditions on the distances than in the case of the 2-SDVRP, the problem can be reduced to a case where each customer has a demand equal to 1 or 2, with a possible reduction on the number of customers. In Section 4 we study the complexity of the k-SDVRP with $k \geq 2$ and prove that, while when $k = 2$ the problem is solvable in polynomial time under specific conditions on the distances, the cases with $k \geq 3$ are NP-hard. Finally, in Section 5 some conclusions are drawn and future developments are presented.

1 Problem definition and reducibility

We define the SDVRP with vehicle capacity $k \in Z^+$, general distances and integer demand for each customer, possibly greater than the vehicle capacity, as the k-SDVRP.

The problem can be represented on a graph $G = (V, E)$ where the set of vertices V represents the depot, vertex 1, and the skips locations, vertices $2, ..., N$. An integer value n_i, indicating the number of skips placed in location i, is associated to each vertex $i \in V - \{1\}$. A cost c_{ij}, representing the distance from vertex i to vertex j, is associated to each edge $(i, j) \in E$. An unlimited number of vehicles, each with a capacity of k skips, is given. Each time a vehicle visits a customer it collects an integer quantity. Each vehicle has to start

from the depot and ends its tour to the depot, visiting a maximum of k vertices. The objective is to minimize the total distance travelled by the vehicles to transport all the skips to the depot.

In the following, we will define for this problem the property of *reducibility*. An instance of the k-SDVRP problem is reducible if it is possible to prove that an optimal solution always exists in which each vertex with k or more skips is served by as many as possible full load direct trips from the depot to the vertex, such that in each direct trip exactly k skips are transported to the depot. Then, for each node, if the number of skips is a multiple of k, no skips remain after the direct trips, while a number less than or equal to $k - 1$ remains in the other cases.

Definition 1 *A $k - SDVRP$ instance is reducible if an optimal solution exists such that* *each vertex that contains k or more skips is served by as many direct trips as possible from the vertex to the depot, transporting k skips in each trip, until less than k skips remain in each vertex.*

When an instance of the problem is reducible, we call *reduced* the instance which is obtained by changing the demand n_i of customer i with $n_i \ (mod\ k)$ and deleting the vertices, and related arcs, when $n_i \ (mod\ k) = 0$. It is obvious that in a reduced problem each vertex is visited only once if $k = 2$, while this may not be the case when $k \geq 3$.

In the next sections we study the problem reducibility when $k = 2$ and $k = 3$. We recall that the distances satisfy the triangle inequality when $c_{ij} \leq c_{ik} + c_{kj}$, $\forall i, j, z$.

The distances satisfy the sharpened triangle inequality, for some $\alpha \in (0, 1]$ when:

$$c_{ij} \leq \alpha(c_{iz} + c_{zj}) \quad \forall i, j, z, i \neq j \neq z. \tag{1}$$

Notice that if the sharpened triangle inequality holds then also the classical triangle inequality holds.

2 The 2-SDVRP

In the 2-SDVRP all the vehicles have a capacity equal to 2 so that they can visit a maximum of 2 customers each.

We analyze the properties of this problem for the different cases where the distances satisfy or do not satisfy the triangle inequality. We start with the particular case in which the distances are symmetrical and satisfy the triangle inequality. We show that in such case the problem is reducible. Then, if the number of skips is even the vertex disappears while one skip remains if the number of skips is odd.

Theorem 1 *The 2-SDVRP with symmetrical distances which satisfy the triangle inequality is reducible.*

Proof. If there are no vertices with 2 or more skips there is nothing to prove. Suppose that at least one such vertex i exists and consider a solution in which 2 skips of i are split into 2 tours. There can be two different cases:

1. In the first tour one skip of i is jointly collected, by a same vehicle, with another skip placed in a different vertex j; in the second tour the other skip of i is jointly collected, by a second vehicle, with another skip placed in vertex z (which can coincide with j). The cost of this solution is:

$$z_1 = C + c_{1i} + c_{ij} + c_{j1} + c_{1i} + c_{iz} + c_{z1}, \tag{2}$$

where C represents the cost of transporting all the other skips to the depot.

Then there exists a solution in which the 2 skips of i are transported together by the same vehicle to the depot, the skips in j and z are joined to form a tour and all the other tours are kept identical; the cost of this solution is:

$$z_2 = C + c_{1i} + c_{i1} + c_{1j} + c_{jz} + c_{z1}. \tag{3}$$

Since the distances are symmetrical and $c_{ij} + c_{iz} \geq c_{jz}$ for the triangle inequality, then $z_2 \leq z_1$.

2. In the first tour one skip of i is jointly collected, by a same vehicle, with another skip placed in a different vertex j; in the second tour the other skip of i is transported alone to the depot. The cost of this solution is:

$$z_1 = C + c_{1i} + c_{ij} + c_{j1} + c_{1i} + c_{i1}. \tag{4}$$

Then there exists a solution in which the 2 skips of i are transported together by the same vehicle to the depot, while the skip in j is transported alone to the depot and all the other tours are kept identical. The cost of this solution is:

$$z_2 = C + c_{1i} + c_{i1} + c_{1j} + c_{j1}. \tag{5}$$

Since the distances are symmetrical and $c_{1i} + c_{ij} \geq c_{1j}$ for the triangle inequality, then $z_2 \leq z_1$.

As the above argument can be used for each pair of skips located in the same vertex, the solution in which full load direct trips are made from each vertex to the depot until only 0 or 1 skip remains in all the vertices is better than or equal to any other solution. Thus the 2-CVRP with symmetrical distances which satisfy the triangle inequality is reducible. □

The same property does not hold for the more general case of asymmetrical distances which satisfy the triangle inequality, as shown by the following example depicted in Figure 1.

Example 1 Let the graph depicted in Figure 1 represent an instance of the 2-SDVRP, where $n_2 = 2$, $n_3 = n_4 = 1$ and the weights on the arcs represent the distances which are asymmetrical and satisfy the triangle inequality. Then, the solution in which one skip of vertex 2 is jointly collected with the skip of vertex 3 in a tour and the remaining skips are collected in another tour has a cost of 8 (as the distances are asymmetrical, the order of visit of the vertices is important and we consider the one that gives the tour with minimum cost), while the solution in which only vertex 2 is visited in one tour and then vertices 3 and 4 in the second tour has a cost of 9. Thus, the first solution is cheaper (it is also the optimal solution) and the problem is not reducible.

Insert here Figure 1

Similarly, it can be easily shown that in the case where the distances are symmetrical and do not satisfy the triangle inequality, the problem is not reducible.

3 The 3-SDVRP

In the 3-SDVRP the capacity of each vehicle is equal to 3 skips. As we have done for the case with capacity equal to 2, we study under which conditions on the distances the problem is reducible.

We first consider the case where the distances satisfy the triangle inequality. In this case, the problem is not reducible even when the distances are symmetrical, as shown by Example 2.

Example 2 Let $G = (V, E)$, $|V| = 5$, be a graph representing a 3-SDVRP where node 1 is the depot, $n_2 = 3$, $n_3 = n_4 = n_5 = 2$ and the distances are symmetrical being all equal to 2 but for the edges incident to node 2 where the distances are equal to 1. Then, the cost of the solution in which three tours are made in which one skip of vertex 2 is joined, respectively, with two skips of vertices 3, 4 and 5 is 12. The cost of all possible solutions that transport the three skips of vertex 2 in one tour and then collect the remaining skips of vertices 3, 4 and 5 in two (or three) tours is 14. Thus, the first solution is cheaper (and it is also the optimal solution) and the 3-SDVRP is not reducible.

From Example 2 it follows that the problem is obviously not reducible for the more general cases where either the distances are asymmetrical or they do not satisfy the triangle inequality. Also for the particular case in which the distances are Euclidean, there exist instances for which the problem is not reducible, as shown by Example 3 reported in Figure 2.

Example 3 Let $\varepsilon > 0$, $k > 0$. Let vertices $2, 3$ and 4 be placed on a circle of radius 2ε while vertex 5 is placed in the center of the circle. Vertices

3 and 4 have both distance equal to ε from vertex 2 and the distance from vertex 2 to the depot is equal to k. Let θ be the angle insisting on the chord $(3, 2)$. With some simple calculus, we can obtain the distances between vertices 3 and 4 and between vertex 3 (or 4) and the depot. Noting that the triangle formed by the vertices $3, 2$ and 5 is isosceles with basis $(3, 2)$ we have $\theta \simeq 28.96°$.

Thus, we get $c_{zt} = 4 * \varepsilon \sin\theta \simeq 1.94\varepsilon$, $c_{hi} = \sqrt{\varepsilon^2 - (\sin\theta * 2\varepsilon)^2} = \frac{1}{4}\varepsilon$. Then, $c_{h1} = k + \frac{1}{4}\varepsilon$ and $c_{z1} = c_{t1} = \sqrt{(\sin\theta * 2\varepsilon)^2 + \left(k + \frac{1}{4}\varepsilon\right)^2} = \sqrt{k^2 + \varepsilon^2 + \frac{1}{2}k\varepsilon}$. The cost of the solution that makes three tours in which 1 skip of vertex 2 is joined, respectively, with 2 skips of vertices $3, 4$ and 5 is $z_1 = 4k + 2c_{z1} + 6\varepsilon$. The cost of the solution that transports 3 skips of vertex 2 in one tour and then 1 skip of vertex 3 (or 4) and 2 skips of vertex 5 in the second tour and 1 skip of vertex 3 (or 4) and 2 skips of 4 (or 3) in the third tour is $z_2 = 3k + 3c_{z1} + 4\varepsilon + c_{zt}$.

By subtracting z_2 from z_1 we obtain:

$$z_1 - z_2 = k + 0.06\varepsilon - \sqrt{k^2 + \varepsilon^2 + \frac{1}{2}k\varepsilon},$$

which is negative for $\varepsilon > -\frac{0.38}{0.9964}k$ and thus for any ε and k. Then the first solution is cheaper than the second one (and it is also the optimal solution) and the 3-SDVRP is not reducible.

Insert here Figure 2

Theorem 2 *The 3-SDVRP is reducible if the distances are symmetrical and satisfy the sharpened triangle inequality (1) with* $\alpha = 2/3$.

Proof. If there are no vertices with 3 or more skips, then there is nothing to prove. Suppose that such a vertex i exists. We first consider the case where each vertex contains at most 3 skips. Let us consider a solution such that the 3 skips of i are split among different tours. We show that a not worse solution exists such that the 3 skips of i are transported together to the depot with a direct trip. There are three main cases to be analyzed.

1. 1 (2) skip(s) of i is (are) transported alone to the depot. Clearly a not worse solution is obtained by moving the 2 (1) remaining skips of i to this trip. The trip that collected these remaining skips in the previous solution does not have to pass through vertex i any more and this causes a reduction of the cost because of the triangle inequality.

2. The 3 skips of vertex i are split in 3 different tours of the kind 1-1-1, i.e., each tour collects 1 skip from i and 1 skip from 2 vertices different from i. Let us call the vertices visited by these tours p, q, j, z, t, and w and $S = \{p, q, j, z, t, w\}$. Each tour visits 3 vertices (not including the depot) and, in each tour, vertex i can be visited as first (or last) vertex (so that the edge $(1, i)$ is included in the tour) or as second vertex (so that the edge $(1, i)$ is not included). Thus, there are 4 solutions to analyze:

   (a) Vertex i is visited as second vertex in each tour. The cost of this solution is, without loss of generality:

   $$z_s = C + c_{1p} + c_{pi} + c_{iq} + c_{q1} + c_{1j} + c_{ji} + c_{iz} + c_{z1} + c_{1t} + c_{ti} + c_{iw} + c_{w1}, \quad (6)$$

   where C represents the cost of transporting all the other skips to the depot. Let us consider a solution that collects the 3 skips of i together, serves the vertices in S and keeps all the other tours identical. There are different solutions of this kind. Let us consider the solution such that:

$$z_d = C + c_{1i} + c_{i1} + c_{1q} + c_{qp} + c_{pj} + c_{j1} + c_{1z} + c_{zt} + c_{tw} + c_{w1}.$$

Let us assume, without loss of generality, that this is the best solution among those that collect the 3 skips of i together. For the sharpened triangle inequality we have:

$$c_{vv'} \leq \alpha(c_{vs'} + c_{s'v'}), \tag{7}$$

$$c_{1i} \leq \alpha(min_{s\in S}\{c_{1s} + c_{si}\}), \tag{8}$$

for all pairs $v, v' \in S$ with $s' = i$ or $s' = 1$. Thus, by applying (7) with $s' = i$ and (8), we obtain:

$$\begin{aligned} z_d \;\leq\; & C + c_{1q} + c_{j1} + c_{1z} + c_{w1} + 2\alpha(min_{s\in S}\{c_{1s} + c_{si}\}) + \alpha(c_{qi} + c_{ij} + c_{iz} + c_{iw}) + \\ & +2\alpha(c_{ip} + c_{it}). \end{aligned} \tag{9}$$

In this solution there are 2 tours that serve the vertices in S (each one serving 3 vertices) in which vertices p and t are visited as second. Let us consider a solution where the vertices visited as second are q and z, both different from p and t:

$$z_d^1 = C + c_{1i} + c_{i1} + c_{1p} + c_{pq} + c_{qj} + c_{j1} + c_{1t} + c_{tz} + c_{zw} + c_{w1}. \tag{10}$$

Since $z_d \leq z_d^1$, by applying (7) and (8), we obtain:

$$\begin{aligned} z_d \;\leq\; & C + c_{1p} + c_{j1} + c_{1t} + c_{w1} + 2\alpha(min_{s\in S}\{c_{1s} + c_{si}\}) + \\ & +\alpha(c_{ps_1} + c_{s_1q} + c_{qs_2} + c_{s_2j} + c_{ts_3} + c_{s_3z} + c_{zs_4} + c_{s_4w}) \end{aligned} \tag{11}$$

where $s_k = i$ or $s_k = 1$, $k = 1, ..., 4$. There are 6 different possibilities of choosing 2 vertices in S, both different from p and t, which must be visited as second in the 2 tours. Let us consider the following 6 solutions of this kind, each solution being considered twice (T_1 is the set of vertices visited, in the order they are listed, in the first tour and T_2 is the set of vertices visited in the second tour):

- $T_1 = \{p, q, j\}$, $T_2 = \{t, z, w\}$, $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, $s_4 = 1$.
- $T_1 = \{p, q, j\}$, $T_2 = \{t, z, w\}$, $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, $s_4 = 1$.
- $T_1 = \{z, q, t\}$, $T_2 = \{p, j, w\}$, $s_1 = i$, $s_2 = 1$, $s_3 = i$, $s_4 = i$.
- $T_1 = \{z, q, t\}$, $T_2 = \{p, j, w\}$, $s_1 = i$, $s_2 = i$, $s_3 = i$, $s_4 = i$.
- $T_1 = \{p, q, z\}$, $T_2 = \{j, w, t\}$, $s_1 = i$, $s_2 = i$, $s_3 = i$, $s_4 = 1$.
- $T_1 = \{p, q, z\}$, $T_2 = \{j, w, t\}$, $s_1 = i$, $s_2 = i$, $s_3 = i$, $s_4 = i$.

- $T_1 = \{t, j, z\}$, $T_2 = \{p, w, q\}$, $s_1 = 1$, $s_2 = i$, $s_3 = 1$, $s_4 = i$.
- $T_1 = \{t, j, z\}$, $T_2 = \{p, w, q\}$, $s_1 = i$, $s_2 = i$, $s_3 = i$, $s_4 = i$.
- $T_1 = \{p, j, q\}$, $T_2 = \{t, z, w\}$, $s_1 = i$, $s_2 = 1$, $s_3 = i$, $s_4 = 1$.
- $T_1 = \{p, j, q\}$, $T_2 = \{t, z, w\}$, $s_1 = i$, $s_2 = 1$, $s_3 = i$, $s_4 = i$.
- $T_1 = \{j, z, t\}$, $T_2 = \{p, w, q\}$, $s_1 = i$, $s_2 = i$, $s_3 = i$, $s_4 = i$.
- $T_1 = \{j, z, t\}$, $T_2 = \{p, w, q\}$, $s_1 = i$, $s_2 = i$, $s_3 = i$, $s_4 = i$.

Note that in solution (10), $T_1 = \{p, q, j\}$ and $T_2 = \{t, z, w\}$. Now, summing up the corresponding inequalities (11) we obtain:

$$\begin{aligned} 12z_d \leq\ & 12C + (12 + 5\alpha)(c_{1p} + c_{1t}) + (6 + 5\alpha)(c_{1z} + c_{1j} + c_{1q} + c_{1w}) + 7\alpha(c_{ip} + c_{it}) + \\ & + 13\alpha(c_{iq} + c_{ij} + c_{iz} + c_{iw}) + 24\alpha(min_{s\in S}\{c_{1s} + c_{si}\}). \end{aligned} \tag{12}$$

Let us now consider a solution where only one of the vertices visited as second is p or t:

$$z_d^2 = C + c_{1i} + c_{i1} + c_{1z} + c_{zp} + c_{pq} + c_{q1} + c_{1t} + c_{tw} + c_{wj} + c_{j1}.$$

As before, since $z_d \leq z_d^2$, by applying (7) with $s' = i$ and (8), we obtain:

$$\begin{aligned} z_d \leq\ & C + c_{1z} + c_{q1} + c_{1t} + c_{j1} + 2\alpha(min_{s\in S}\{c_{1s} + c_{si}\}) + \alpha(c_{iz} + c_{iq} + c_{it} + c_{ij}) + \\ & + 2\alpha(c_{ip} + c_{iw}). \end{aligned} \tag{13}$$

There are 8 possibilities of choosing two vertices in S which must be visited as second, one of which is p or t. Summing up the corresponding inequalities (13) where s' is always equal to i we obtain:

$$\begin{aligned} 8z_d \leq\ & 8C + 4(c_{1p} + c_{1t}) + 6(c_{1z} + c_{1q} + c_{1w} + c_{1j}) + 12\alpha(c_{ip} + c_{it}) + \\ & + 10\alpha(c_{iz} + c_{iq} + c_{iw} + c_{ij}) + 16\alpha(min_{s\in S}\{c_{1s} + c_{si}\}). \end{aligned} \tag{14}$$

Summing up inequality (12) with the double of inequalities (9) and (14) we obtain:

$$30z_d \leq 30C + (20 + 5\alpha)\sum_{s\in S} c_{1s} + 35\alpha \sum_{s\in S} c_{is} + 60\alpha(min_{s\in S}\{c_{1s} + c_{si}\}).$$

Since $|S| = 6$, we know that:

$$min_{s\in S}\{c_{1s} + c_{si}\} \leq \frac{1}{6}\sum_{s\in S} c_{1s} + \frac{1}{6}\sum_{s\in S} c_{is}$$

and thus we obtain:

$$30z_d \leq 30C + (20+5\alpha)\sum_{s\in S} c_{1s} + 35\alpha \sum_{s\in S} c_{is} +$$
$$+60\alpha(\frac{1}{6}\sum_{s\in S} c_{1s} + \frac{1}{6}\sum_{s\in S} c_{is}).$$

Now, recalling the value of z_s given in (6), if we find a value of α for which:

$$30z_s = 30C + 30\sum_{s\in S} c_{1s} + 30\sum_{s\in S} c_{is} \geq 30C + (20+5\alpha)\sum_{s\in S} c_{1s} +$$
$$+35\alpha\sum_{s\in S} c_{is} + 60\alpha(\frac{1}{6}\sum_{s\in S} c_{1s} + \frac{1}{6}\sum_{s\in S} c_{is})$$

i.e.:

$$(2-3\alpha)\sum_{s\in S} c_{1s} + (6-9\alpha)\sum_{s\in S} c_{is} \geq 0 \tag{15}$$

then $z_d \leq z_s$. Since for $\alpha \leq \frac{2}{3}$ inequality (15) is satisfied, then for $\alpha = \frac{2}{3}$ an optimal solution exists such that the 3 skips of i are transported together to the depot with a direct trip.

(b) Vertex i is visited as second vertex in two tours. The cost of this solution is, without loss of generality:

$$z_s = C + c_{1p} + c_{pi} + c_{iq} + c_{q1} + c_{1j} + c_{ji} + c_{iz} + c_{z1} + c_{1i} + c_{it} + c_{tw} + c_{w1}.$$

Let us assume that the best solution that collects the 3 skips of i together, serves the vertices in S and keeps all the other tours identical is the following:

$$z_d = C + c_{1i} + c_{i1} + c_{1p} + c_{pq} + c_{qz} + c_{z1} + c_{1j} + c_{jt} + c_{tw} + c_{w1}.$$

Following the same procedure as for case 2.a we can show that, since the distances satisfy the sharpened triangle inequality with $\alpha = \frac{2}{3}$, $z_d \leq z_s$.

(c) Vertex i is visited as second vertex in one tour only. The cost of this solution is, without loss of generality:

$$z_s = C + c_{1i} + c_{ip} + c_{pq} + c_{q1} + c_{1i} + c_{ij} + c_{jt} + c_{t1} + c_{1z} + c_{zi} + c_{iw} + c_{w1}.$$

Let us consider the following solution that collects the 3 skips of i together:

$$z_d = C + c_{1i} + c_{i1} + c_{1q} + c_{qp} + c_{pz} + c_{z1} + c_{1t} + c_{tj} + c_{jw} + c_{w1}.$$

Since the distances satisfy the sharpened triangle inequality with $\alpha = \frac{2}{3}$, we have:

$$c_{pz} \leq \frac{2}{3}(c_{ip} + c_{iz}),$$
$$c_{jw} \leq \frac{2}{3}(c_{ji} + c_{iw}).$$

Thus, $z_d \leq z_s$.

(d) Vertex i is visited as first (or last) vertex in each tour. The cost of this solution is, without loss of generality:

$$z_s = C + c_{1i} + c_{ip} + c_{pq} + c_{q1} + c_{1i} + c_{it} + c_{tj} + c_{j1} + c_{1i} + c_{iz} + c_{zw} + c_{w1}.$$

Let us assume that the best solution that collects the 3 skips of i together, serves the vertices in S and keeps all the other tours identical is the following:

$$z_d = C + c_{1i} + c_{i1} + c_{1q} + c_{qp} + c_{pt} + c_{t1} + c_{1j} + c_{jz} + c_{zw} + c_{w1}.$$

Following the same procedure as for case 2.a we can show that, since the distances satisfy the sharpened triangle inequality with $\alpha = \frac{2}{3}$, $z_d \leq z_s$.

3. In the first tour, a vehicle collects 1 skip from i, 1 skip from another vertex j and 1 skip from a third vertex z. In the second tour, a vehicle collects 2 skips of i and 1 skip of another vertex t. We define $S' = \{j, z, t\}$. We have 2 cases:

(a) In the first tour vertex i is visited as second vertex. The cost of this solution is, without loss of generality:

$$z_s = C + c_{1j} + c_{ji} + c_{iz} + c_{z1} + c_{1i} + c_{it} + c_{t1}.$$

Let us assume that the best solution that collects the 3 skips of i together, serves the vertices in S' and keeps all the other tours identical is the following:

$$z_d = C + c_{1i} + c_{i1} + c_{1t} + c_{tj} + c_{jz} + c_{z1}.$$

Following the same procedure as for case 2.a we can show that, since the distances satisfy the sharpened triangle inequality with $\alpha = \frac{2}{3}$, $z_d \leq z_s$.

(b) In the first tour vertex i is visited as first (or last) vertex. The cost of this solution is, without loss of generality:

$$z_s = C + c_{1i} + c_{ij} + c_{jz} + c_{z1} + c_{1i} + c_{it} + c_{t1}.$$

Let us consider the following solution that collects the 3 skips of i together:

$$z_d = C + c_{1i} + c_{i1} + c_{1t} + c_{tj} + c_{jz} + c_{z1}.$$

Since the distances satisfy the sharpened triangle inequality with $\alpha = \frac{2}{3}$, we have:

$$c_{tj} \leq \frac{2}{3}(c_{it} + c_{ij})$$

thus, $z_d \leq z_s$.

The cases where the solution contains at least one tour which collects 1 skip from i and 2 skips from another vertex can be proved with the proof of cases 2 or 3 where two vertices of S or S' coincide (thus, their distance is set to 0). Consequently, in the case where there is only one vertex which contains 3 skips, if the distances satisfy the sharpened triangle inequality with $\alpha = \frac{2}{3}$ the problem is reducible.

If there is more than one vertex with 3 skips we can iterate this procedure on all these vertices. Finally, if there are vertices in which there are more than three skips, we can split them to obtain vertices with at most 3 skips and iterating the above procedure we obtain a reduced solution which is not worse than any other solution. Thus, if $\alpha = 2/3$ the 3-SDVRP with symmetrical distances is reducible. □

The result of Theorem 2 cannot be strengthened by any larger value of α as it is shown by the following example.

Example 4 Consider an instance for the 3-SDVRP with the depot and four vertices, with $n_2 = 3$, $n_3 = n_4 = n_5 = 2$ and distances equal to $\frac{4}{3} + \varepsilon$, $\varepsilon > 0$, for all edges but the ones insisting on vertex 2 which are set to 1. The distances satisfy the sharpened triangle inequality with $\alpha = 2/3 + \varepsilon/2$ and do not for any smaller value of α. The cost of the solution that makes three tours in which 1 skip of vertex 2 is joined with 2 skips of vertices 3, 4 and 5, respectively is $10 + 3\varepsilon$. The cost of each of the four solutions that transport the three skips of vertex 2 together to the depot and then collect the skips of vertices 3, 4 and 5 is $10 + 6\varepsilon$.

Thus, for any $\varepsilon > 0$, it is not optimal to make direct trips from vertex 2 and the problem is not reducible.

Note that, contrary to the case of the 2-SDVRP, in a reduced instance of the 3-SDVRP a vertex may be visited more than once, as shown by the following example.

Example 5 Consider a reduced 3-SDVRP problem with four vertices, the depot (vertex 1) and the vertices $2, 3$ and 4 such that $n_2 = n_3 = n_4 = 2$, where the distances are all equal to 4/3 but those on the edges incident to node 2 which are equal to 1. Then the optimal solution consists of two tours in which vertices 3 and 4 are visited only once while vertex 2 twice.

For the case where k is greater than 3, the value of α for which the problem is reducible is at most 2/3 as it is always possible to build an instance similar to Example 4 with the same number of vertices, the same cost for each edge, $n_2 = k$, $n_3 = n_4 = k - 1$, $n_5 = 2$.

4 Computational complexity of the k-SDVRP

In this section we study the complexity of the k-SDVRP.

If $k = 2$, when the distances are symmetrical and satisfy the triangle inequality, from Theorem 1, the problem is reducible and, since the sum of the three smallest customer demands exceeds the capacity of the vehicles, the problem is solvable with a matching algorithm (Christofides, 1985).

Remark 1 *The 2-SDVRP with symmetrical distances that satisfy the triangle inequality can be solved in $O(N^3)$ time.*

In (Dror, Laporte and Trudeau, 1994) and (Dror and Trudeau, 1990) it is shown that in the SDVRP, for every value of the vehicle capacity, if the distances satisfy the triangle inequality then there always exists an optimal solution where no pair of tours has more than 1 vertex in common, because this will generate a t-split cycle (for a definition of the t-split cycle, see (Dror and Trudeau, 1990)). We now show that, when $k = 2$, this result holds also when the distances are symmetrical and do not satisfy the triangle inequality. This property makes the problem solvable in polynomial time when the distances are symmetrical or satisfy the triangle inequality.

Theorem 3 *If $c_{1i} = c_{i1}$ $i = 2, ..., N$ or c_{1i} and c_{i1} are the smallest cost to and from the depot $i = 2, ..., N$, then the 2-SDVRP can be solved in $O(N^6)$ time.*

Proof. We prove that, for each pair of vertices i, $j = 2, ..., N$, in the optimal solution there is no more than 1 tour that joins 1 skip of i with 1 skip of j. The reason is that the cost of 2 tours that join 1 skip of i with 1 skip of j is greater than (or equal to) the cost of transporting 2 skips of i and 2 skips of j with direct trips to the depot.

The cost of 2 tours that join 1 skip of i with 1 skip of j is, without loss of generality:

$$c_s = 2c_{1i} + 2c_{ij} + 2c_{j1}$$

while the cost of transporting 2 skips of i and 2 skips of j with direct trips to the depot is:

$$c_d = c_{1i} + c_{i1} + c_{1j} + c_{j1}.$$

Obviously, if $c_{1i} = c_{i1}$ or c_{1i} and c_{i1} are the smallest cost to and from the depot, $i = 2, ..., N$, then the latter solution is always better than (or equal to) the former one. Thus for each pair of vertices i and j, there will be no more than 1 tour that joins 1 skip of i with 1 skip of j. Consequently, for each vertex i, the corresponding skips will be joined with at most $N - 2$ skips of other vertices, one for each $j = 2, ..., N$, $j \neq i$, and we obtain that at most $N - 2$ skips of each vertex will be joined with skips of other vertices, while for the remaining it is optimal to do direct trips to the depot. Thus, we obtain a problem where $n_i \leq N - 2$, $i = 2, ..., N$. We can now transform this problem by creating for each vertex containing n_i skips n_i identical vertices containing 1 skip each and setting at 0 the distances among them. In this case the problem can be solved with an $O(((N - 2) * N)^3) = O(N^6)$ matching algorithm. □

Corollary 1 *If the distances are symmetrical, then the 2-SDVRP can be solved in $O(N^6)$ time.*

Corollary 2 *If the distances satisfy the triangle inequality, then the 2-SDVRP can be solved in $O(N^6)$ time.*

Theorem 4 *If neither the distances are symmetrical nor satisfy the triangle inequality, then the 2-SDVRP can be solved in $O((\sum_{i=2}^{N} n_i)^3)$ time.*

Proof. We can transform the problem by creating for each vertex containing n_i skips n_i identical vertices containing 1 skip each and setting at 0 the distances among them. In this case the problem can be solved with an $O((\sum_{i=2}^{N} n_i)^3)$ matching algorithm which is a pseudo-polynomial time algorithm. □

We now prove that, if $k \geq 3$, the decision version of the k-SDVRP is NP-complete.

INSTANCE: Graph $G = (V, E)$, $V = \{1, ..., N\}$, distances $c_{ij} \in Z^+$, $(i, j) \in E$, weights $n_i \in Z^+$, $i \in V - \{1\}$ (which represent the number of skips in each vertex except the depot), capacity $k \in Z^+$ and positive integer B.

QUESTION: Is there a set of tours starting and ending at the depot, transporting all the skips to the depot, such that in each tour the sum of the skips collected is not greater than k and the total cost of the tours, given by the sum of the distances travelled in each tour, is not greater than B?

Theorem 5 *The decision version of the k-SDVRP where each customer has unitary demand, with symmetrical distances that satisfy the sharpened triangle inequality with $1/2 < \alpha \leq 1$ is NP-complete for $k \geq 3$.*

Proof. We consider a variant of the *Partition into Isomorphic Subgraphs Problem* (PIIS) which is stated as follows (Garey and Johnson, 1979):

INSTANCE: Graphs $\bar{G} = (\bar{V}, \bar{E})$ and $H = (V', E')$ with $\left|\bar{V}\right| = q\,|V'|$ for some $q \in Z^+$.

QUESTION: Can the vertices of $\bar{G}$ be partitioned into q disjoint sets $V_1, V_2, ..., V_q$ such that, for $1 \leq i \leq q$, the subgraph of $\bar{G}$ induced by V_i contains a subgraph isomorphic to H?

It has been shown (Garey and Johnson, 1979) that this variant of the PIIS remains NP-complete for any fixed H which contains a connected component of three or more vertices. In the following we show a polynomial transformation from this variant of the PIIS, with H a path of k vertices, $k \geq 3$ (from now on simply PIIS-path problem), to the k-SDVRP. Let us consider an instance of the PIIS-path problem. We now construct the corresponding instance of the k-SDVRP. Let G be a complete graph with $V = \bar{V} \cup \{1\}$, where vertex 1 is the depot. Thus $N = qk + 1$. We take $n_i = 1$, $i \in \bar{V}$. The distances are symmetrical and we set, for $i, j \in \bar{V}$ and $0 < \varepsilon \leq 2\alpha - 1$,

$$c_{ij} = \begin{cases} 1 \text{ if } (i,j) \in \bar{E} \\ 1 + \varepsilon \text{ if } (i,j) \notin \bar{E}. \end{cases}$$

Moreover, we set $c_{1i} = C = 1 + \varepsilon$, $i \in \bar{V}$. Clearly, c_{ij}, $i, j \in V$, satisfy the sharpened triangle inequality with $1/2 < \alpha \leq 1$. Finally, we take $B = (k - 1 + 2C)q$. The minimum number of tours to serve the vertices in $\bar{V}$ is q, thus the cost due the edges that are incident to vertex 1 in the solution of the k-SDVRP is at least $2Cq$. The minimum number of edges to be taken to serve the vertices in $\bar{V}$ is $q(k + 1)$. Thus, the minimum cost to serve the vertices in $\bar{V}$ is $(k - 1 + 2C)q = B$. The solutions which make more than q tours, e.g. $q + d$ tours, $d \in Z^+$, have a cost of $2C(q + d)$ for the edges that are incident to the depot and a cost of at least $q(k - 1) - 2d$ for the remaining edges (since the total minimum number of edges is $q(k + 1)$). Thus the total cost of these solutions is greater than or equal to $2C(q + d) + q(k - 1) - 2d = q(k - 1 + 2C) + 2d(C - 1) > B$. Thus, we need only to consider solutions with q tours, where each tour visits k vertices (if a tour visits less than k vertices then there exists a tour which visits more than k vertices and this violates the capacity constraint). An instance of the k-SDVRP has 'yes' answer if and only if there exist q paths, which visit k vertices each, cover all the vertices in $\bar{V}$ and have total cost $B' = B - 2Cq = (k - 1)q$. Such a solution exists if and only if each of the q paths has cost lower than or equal to $k - 1$. Since all the edges of G have cost greater than or equal to 1 each path must have a cost of $k - 1$, that is all edges in each path must have cost equal to 1. We can now see that the instance of the PIIS-path problem has 'yes' answer if and only if the corresponding instance of the k-SDVRP has 'yes' answer. If the PIIS-path problem has 'yes' answer, then there exists a partition of $\bar{V}$ into q sets such that the subgraph of $\bar{G}$ induced by each set contains a path of k vertices. Since these paths identify q paths of cost $k - 1$ each in G, the k-SDVRP has 'yes' answer. Viceversa, if the k-SDVRP has 'yes' answer, then there exist q paths of $k - 1$ edges of cost 1 each. This implies that these edges are edges of $\bar{E}$ and then the PIIS-path problem has 'yes' answer. Since this is a polynomial transformation, the decision version of the k-SDVRP is NP-complete. □

Remark 2 *Since the k-SDVRP with unitary demands is a particular case of the CVRP, the CVRP with unitary demands and symmetrical distances that satisfy the sharpened triangle inequality with $1/2 < \alpha \leq 1$ is NP-hard when the capacity of the vehicles is greater or equal to 3.*

Obviously, all the more general cases, for instance the tipically considered case of distances which satisfy the triangle inequality, are NP-hard.

5 Conclusions and future developments

In this paper we have analyzed the k-SDVRP which is a vehicle routing problem with general distances and multiple customer visits, where the capacity of the vehicles is $k \geq 2$, $k \in Z^+$ and each customer has a discrete demand possibly larger than the vehicle capacity. We have shown that for $k \geq 3$ the problem is NP-hard, while if $k = 2$, when the distances are symmetrical or satisfy the triangle inequality, it is solvable in polynomial time. For $k = 2$ and $k = 3$, we have proved that this problem is reducible in polynomial time to a new problem where each customer has a demand of less than k skips only under restricted conditions on the distances. Clearly, this reduction causes a remarkable simplification of the problem with also a possible reduction on the number of customers. We have also shown that, while if $k = 2$ the reduced problem becomes a particular case of the VRP since each customer must be visited only once, if $k = 3$ even in the reduced problem a customer can be visited more than once in the optimal solution and thus it is not a particular case of the VRP. A still open question concerns the existence of some conditions that allow to reduce the problem in the case of asymmetrical distances for $k \geq 2$. For the case $k > 3$ we have shown that the problem, in the case of symmetrical distances, may be reducible only if the sharpened triangle inequality holds with α equal to at most 2/3. An open problem is to find the precise conditions for the reducibility.

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References

[1] K. Altinkemer, B. Gavish, "Parallel savings based heuristics for the delivery problem", *Ops. Res.* 39, 456–469 (1991).

[2] K. Altinkemer, B. Gavish, "Heuristics for delivery problems with constant error guarantees", *Transp. Science* 24, 295–297 (1990).

[3] K. Altinkemer, B. Gavish, "Heuristics for unequal weight delivery problems with a fixed error guarantees", *Opns. Res. Lett.* 6, 149–158 (1987).

[4] M. Ball, L. Bodin, R. Baldacci, A. Mingozzi, "The rollon-rolloff vehicle routing problem", *Transp. Science* 34, 271–288 (2000).

[5] J.M. Belenguer, M.C. Martinez, E. Mota, "A lower bound for the split delivery vehicle routing problem", *Ops. Res.* 48, 801-810 (2000).

[6] N. Christofides, "Vehicle routing", in *The Traveling Salesman Problem*, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnoy Kan, D.B. Shmoys (eds), 431-448, John Wiley, Chichester, 1985.

[7] N. Christofides, A. Mingozzi, P. Toth, "The vehicle routing problem", in *Combinatorial Optimization*, N. Christofides, A. Mingozzi, P. Toth, C. Sandi, (eds.) John Wiley & Sons, Chichester, 1979.

[8] L. De Meulemeester, G. Laporte, F.V. Louveaux, F. Semet, "Optimal scheduling of skip collections and deliveries", *J. of Operational Res. Society* 48, 57–64 (1997).

[9] M. Dror, G. Laporte, P. Trudeau, "Vehicle routing with split deliveries", *Discrete Appl. Math.* 50, 239-254 (1994).

[10] M. Dror, P. Trudeau, "Savings by split delivery routing", *Transp. Science* 23, 141-145 (1989).

[11] M. Dror, P. Trudeau, "Split delivery routing", *Naval Res. Logist.* 37, 383-402 (1990).

[12] P.W. Frizzell, J.W. Giffin, "The split delivery vehicle scheduling problem with time windows and grid network distances", *Computers Ops. Res.* 22, 655-667 (1995).

[13] M.R. Garey, D.S. Johnson, "*Computers and intractability: a guide to the theory of NP-completeness*", W. H. Freeman, New York, NY (1979).

[14] B.L. Golden, A.A. Assad, "*Vehicle routing: methods and studies*", North-Holland, Amsterdam (1988).

[15] P.A. Mullaseril, M. Dror, J. Leung, "Split-delivery routing in livestock feed distribution", *J. of Operational Res. Society* 48, 107-116 (1997).

[16] G. Sierksma, G.A. Tijssen, "Routing helicopters for crew exchanges on off-shore locations", *Ann. Oper. Res.* 76, 261-286 (1998).