Bayesian Estimation of Structural Models

III: More on Bayesian Estimation of DSGE models

Gianni Amisano

University of Brescia

European Central Bank

amisano@eco.unibs.it

Vienna, 25/04/08

Contents

1 Perturbation methods

Draws on S. Schmitt-Grohè (2005)

DSGE can be written as a system of nonlinear expectation stochastic difference equation

$$
\begin{aligned}
E_t f(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t) &= \underset{(n\times 1)}{\mathbf{0}}, n = n_x + n_y & (1)\\
\underset{(n_x\times 1)}{\mathbf{x}_t} &= \text{state variables (exog or predetermined)} & (2)\\
\underset{(n_y\times 1)}{\mathbf{y}_t} &= \text{control variables (endogs)} & (3)
\end{aligned}
$$

Initial condition $\mathbf{x}_0 + NPG$

Perturbation method: consider solution as function of state vector x_t and of parameter σ

$$\mathbf{x}_{t+1} = H(\mathbf{x}_t, \sigma) + \sigma \underline{\mathbf{B}} \mathbf{w}_{t+1}, \underline{\mathbf{B}} = \begin{bmatrix} [\mathbf{0}] \\ \underline{\mathbf{B}}_2 \end{bmatrix} \quad (4)$$

$$\mathbf{y}_t = G(\mathbf{x}_t, \sigma) \quad (5)$$

In the example seen yesterday we have

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \sigma_a & 0 \\ 0 & \sigma_d \end{bmatrix}, \sigma = \sigma_a + \sigma_d, \underline{\mathbf{B}} = \begin{bmatrix} 0 & 0 \\ \frac{\sigma_a}{\sigma_a + \sigma_d} & 0 \\ 0 & \frac{\sigma_d}{\sigma_a + \sigma_d} \end{bmatrix}$$

approximate with (linear) Taylor expansion around $(\overline{\mathbf{x}}, \overline{\sigma})$

$$
\begin{aligned}
H(\mathbf{x}_t, \sigma) &\approx H(\overline{\mathbf{x}}, \overline{\sigma}) + \mathbf{H}_\sigma(\sigma - \overline{\sigma}) + \mathbf{H}_\mathbf{x}(\mathbf{x}_t - \overline{\mathbf{x}}) && (6) \\
G(\mathbf{x}_t, \sigma) &\approx G(\overline{\mathbf{x}}, \overline{\sigma}) + \mathbf{G}_\sigma(\sigma - \overline{\sigma}) + \mathbf{G}_\mathbf{x}(\mathbf{x}_t - \overline{\mathbf{x}}) && (7)
\end{aligned}
$$

Here unknowns are the derivatives.

To find them plug system solution into (1) to obtain

$$
F(\mathbf{x}, \sigma) = E_t f \left\{ G[H(\mathbf{x}, \sigma) + \sigma \underline{\mathbf{B}} \mathbf{w}^{'}], G(\mathbf{x}, \sigma), H(\mathbf{x}, \sigma) + \sigma \underline{\mathbf{B}} \mathbf{w}^{'}, \mathbf{x} \right\} \quad (8)
$$

Since $F(\mathbf{x}, \sigma)$ must be zero for any $\mathbf{x}, \sigma$, all its derivatives must be zero too

$$
F_{[\mathbf{x}]^k, [\sigma]^j} = [\mathbf{0}] \quad (9)
$$

Approximate around non stochastic steady state (NSSS)

$$\mathbf{x}_t = \overline{\mathbf{x}}, \sigma = 0 \tag{10}$$

such that

$$f(\overline{\mathbf{y}}, \overline{\mathbf{y}}, \overline{\mathbf{x}}, \overline{\mathbf{x}}) = 0 \tag{11}$$

$$\overline{\mathbf{y}} = G(\overline{\mathbf{x}}, 0) \tag{12}$$

$$\overline{\mathbf{x}} = H(\overline{\mathbf{x}}, 0) \tag{13}$$

Differentiate F wrt σ

$$\begin{aligned} \mathbf{F}_\sigma &= \\ E_t\left\{\mathbf{f}_{y'}[\mathbf{G}_x(\mathbf{H}_\sigma + \underline{\mathbf{B}}\mathbf{w}') + \mathbf{G}_\sigma] + \mathbf{f}_y\mathbf{G}_\sigma + \mathbf{f}_{x'}(\mathbf{H}_\sigma + \underline{\mathbf{B}}\mathbf{w}')\right\} &= \\ \mathbf{f}_{y'}[\mathbf{G}_x\mathbf{H}_\sigma + \mathbf{G}_\sigma] + \mathbf{f}_y\mathbf{G}_\sigma + \mathbf{f}_{x'}\mathbf{H}_\sigma &= \underset{(n\times 1)}{\mathbf{0}} \Rightarrow \\ \begin{bmatrix} \underset{(n\times n_x)}{(\mathbf{f}_{y'}\mathbf{G}_x + \mathbf{f}_{x'})} & \underset{(n\times n_y)}{(\mathbf{f}_{y'} + \mathbf{f}_y)} \end{bmatrix} \begin{bmatrix} \underset{(n_x\times 1)}{\mathbf{H}_\sigma} \\ \underset{(n_y\times 1)}{\mathbf{G}_\sigma} \end{bmatrix} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{14}$$

Homogeneous system: will have unique solution

$$\begin{bmatrix} \underset{(n_x\times 1)}{\mathbf{H}_\sigma} \\ \underset{(n_y\times 1)}{\mathbf{G}_\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{15}$$

Remarkable result

- approx are simpler than we thought! (good news)
- certainty equivalence principle holds up to FO effects (good news)
- linear approx gives no role to scale of uncertainty (risk premia, welfare analysis) (bad news?)

To find $\mathbf{G}_x$ and $\mathbf{H}_x$, differentiate F wrt $\mathbf{x}$

$$
\begin{aligned}
\mathbf{F}_x &= E_t\left\{\mathbf{f}_{y'}[\mathbf{G}_x\mathbf{H}_x] + \mathbf{f}_y\mathbf{G}_x + \mathbf{f}_{x'}\mathbf{H}_x + \mathbf{f}_x\right\} = \\
&= \mathbf{f}_{y'}\mathbf{G}_x\mathbf{H}_x + \mathbf{f}_y\mathbf{G}_x + \mathbf{f}_{x'}\mathbf{H}_x + \mathbf{f}_x = \underset{(n\times 1)}{\mathbf{0}} \Rightarrow \\
\begin{bmatrix} \mathbf{f}_{x'} & \mathbf{f}_{y'} \end{bmatrix}\begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{G}_x \end{bmatrix}\mathbf{H}_x &= -\begin{bmatrix} \mathbf{f}_x & \mathbf{f}_y \end{bmatrix}\begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{G}_x \end{bmatrix} \Rightarrow \qquad (16) \\
\underset{(n\times n)}{\mathbf{A}}\underset{(n\times n_x)}{\mathbf{Z}}\underset{(n_x\times n_x)}{\mathbf{\Lambda}} &= \underset{(n\times n)}{\mathbf{B}}\underset{(n\times n_x)}{\mathbf{Z}}, \text{(GENERALISED EIG)} \qquad (17) \\
\mathbf{A} &= \begin{bmatrix} \mathbf{f}_{x'} & \mathbf{f}_{y'} \end{bmatrix}, \mathbf{B} = -\begin{bmatrix} \mathbf{f}_x & \mathbf{f}_y \end{bmatrix}, \qquad (18) \\
\mathbf{Z} &= \begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{G}_x \end{bmatrix}\mathbf{P}, \qquad (19) \\
\mathbf{H}_x\mathbf{P} &= \mathbf{P\Lambda} \qquad (20)
\end{aligned}
$$

Non linear system. Can be solved by finding generalised eigenvalues-vectors

$$
\begin{aligned}
&\mathbf{A}[\underset{(n\times n_x)}{\mathbf{V}_1}\ \underset{(n\times n_y)}{\mathbf{V}_2}]\begin{bmatrix} \underset{(n_x\times n_x)}{\mathbf{D}_{11}} & [\mathbf{0}] \\ [\mathbf{0}] & \underset{(n_y\times n_y)}{\mathbf{D}_{22}} \end{bmatrix} \\
&= \mathbf{B}[\mathbf{V}_1\mathbf{V}_2] \Rightarrow && (21)
\end{aligned}
$$

$$
\mathbf{A}\mathbf{V}_1\mathbf{D}_{11} = \mathbf{B}\mathbf{V}_1 \Rightarrow \quad (22)
$$

$$
\begin{bmatrix} \mathbf{I}_{n_x} \\ \mathbf{G}_x \end{bmatrix}\mathbf{P} = \mathbf{V}_1 = \begin{bmatrix} \mathbf{V}_{11} \\ \mathbf{V}_{12} \end{bmatrix}, \quad (23)
$$

$$
\mathbf{\Lambda} = \mathbf{D}_{11} \Rightarrow \quad (24)
$$

$$
\mathbf{G}_x = \mathbf{V}_{12}\mathbf{V}_{11}^{-1}, \quad (25)
$$

$$
\mathbf{H}_x = \mathbf{V}_{11}\mathbf{D}_{11}\mathbf{V}_{11}^{-1} \quad (26)
$$

Remarks

- need obtain (numerically) $\mathbf{f}_x, \mathbf{f}_{x'}, \mathbf{f}_y, \mathbf{f}_{y'}$: use Matlab symbolic Toolbox

- can use alternative Schur decomposition

- can use higher order approximations (Fernandex and Villaverde, An and Schorfheide, Amisano and Tristani): second order approximation requires solution of additional linear system.

2 Another model: Amisano and Tristani (2007)

- Standard Woodford set-up with monopolistically competitive firms and Calvo pricing. Probability of not being able to readjust prices: ζ.

- Two equally standard modifications to improve macroeconomic fit (Christiano, Eichenbaum & Evans, 2005, Smets & Wouters, 2003):

- (Internal) habit persistence related to 1 lag of (parameter: h).

- (Partial) inflation indexation to lagged inflation (parameter: ι).

- Structural hocks: technology and cost-push (via income tax). Persistence parameters: ρ_a and ρ_τ.

2.1 Main ingredients

- $U(C_t, H_t, L_t) = \frac{(C_t - hC_{t-1})^{1-\gamma}}{1-\gamma} - \int_0^1 \chi L_t(i)^\phi \, di$

- heterogeneous goods $C = \left(\int_0^1 C(i)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$

- MC

- no capital

- Following Steinsson (2003)stochastic income tax $\frac{\tau_t}{1+\tau_t}$

$$\log \tau_t = (1 - \rho_\tau) \overline{\tau} + \rho_\tau \log \tau_{t-1} + \varepsilon_t^\tau, \varepsilon_t^\tau \sim N\left(0, \sigma_\tau^2\right).$$

- $Y_t(i) = A_t L(i)^{\alpha}$, $A_t = A_{t-1}^{\rho_a} e^{\varepsilon_t^a}$

2.2 Price stickiness

- Calvo (1983) contracts :constant probability ζ of being unable to change their price at each time t.

- rule of thumb partial indexation:

$$P_t^i = P_{t-1}^i \left(\overline{\Pi}\right)^{1-\iota} (\Pi_{t-1})^{\iota}$$

- (if integrated inflation target, steady state inflation is not defined$\Leftrightarrow \iota = 1$).

2.3 Two variants of the policy rule

Two different policy rules: with stationary or integrated inflation objective:

- Model 1 (AT1)

$$\begin{aligned} i_t &= (1-\rho_I)(\overline{\pi} - \ln\beta) + \psi_\pi (\pi_t - \pi_t^*) + \\ &\quad + \psi_y (y_t - y_t^n) + \rho_I i_{t-1} + \varepsilon_t^i \\ \pi_t^* &= (1-\rho_\pi)\overline{\pi} + \rho_\pi \pi_{t-1}^* + \varepsilon_t^{\pi^*} \end{aligned}$$

- Model 2 (AT2)

$$\begin{aligned} i_t &= (1-\rho_I)\Big((\pi_t^* - \ln\beta) + \psi_\pi (\pi_t - \pi_t^*) + \psi_y (y_t - y_t^n)\Big) + \\ &\quad + \rho_I i_{t-1} + \varepsilon_t^i \\ \pi_t^* &= \pi_{t-1}^* + \varepsilon_t^{\pi^*} \end{aligned}$$

2.4 Variables in the model

Variables

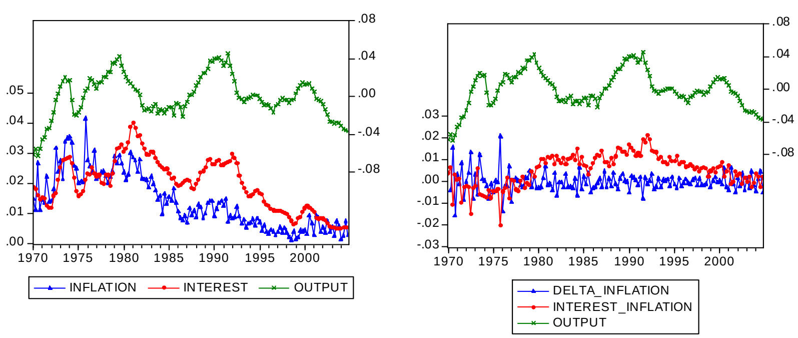
$$\mathbf{x}_t = \begin{bmatrix} \pi_{t-1} \\ y^n_{t-1} \\ y_{t-1} \\ i_{t-1} \\ A_t \\ \pi^*_t \\ \tau_t \\ \varepsilon^i_t \end{bmatrix}, \mathbf{y}^o_t = \begin{bmatrix} \pi_t \\ i_t \\ y_t \end{bmatrix}, \mathbf{w}_t = \begin{bmatrix} \text{tech sh} \\ \text{target sh} \\ \text{policy sh} \\ \text{tax sh} \end{bmatrix}$$

2.5 Results

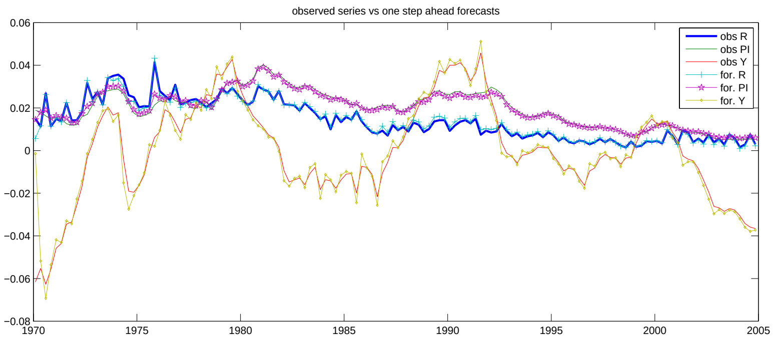
- Observables: Y, R, PI taken from the Area Wide Model database (see Fagan, Henry and Mestre, 2005).

- Raw data, only output is detrended.

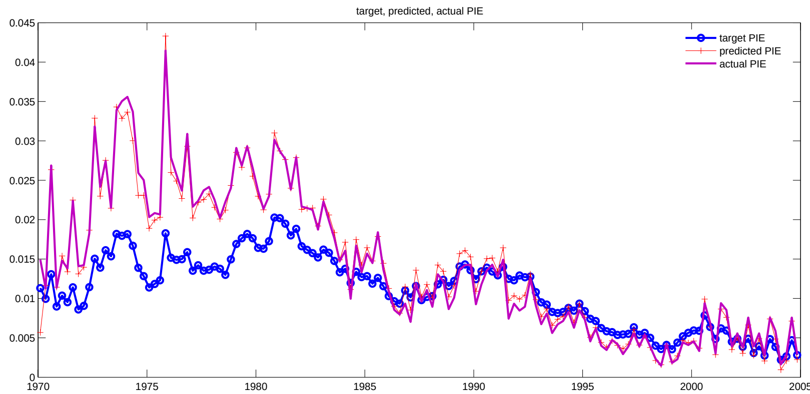
- The estimation period runs from 1970Q1 to 2004Q4.



2.5.1 overall fit



2.5.2 Results: the latent target



2.5.3 Results: Parameter estimates

	prior distrib.		**post. linear m.**	
parameter	**mean**	**sd**	**mean**	**sd**
γ	1.999	0.704	3.275	0.868
h	0.699	0.137	0.476	0.061
θ	7.989	2.635	6.870	2.274
ζ	0.600	0.147	0.403	0.071
ι	0.669	0.176	0.077	0.038
ψ_π	2.000	0.182	1.949	0.169
ψ_y	0.050	0.035	0.043	0.032
ρ_i	0.801	0.100	0.894	0.015
ρ_a	0.900	0.091	0.998	0.002
ρ_π	0.899	0.091	0.990	0.007
$100 \times \sigma_a$	0.334	0.150	1.351	0.164
$100 \times \sigma_\pi$	0.125	0.055	0.135	0.019
π	1.005	0.003	1.005	0.003

- posterior distributions with mean far from prior mean. As an example, RRA posterior around 4.0, prior around 2.0.

- Not all parameters, have tighter posterior distributions : γ (RRA), ϕ (labor disutility), the policy rule parameters ψ_π and ψ_y, and the parameters describing the properties of the tax shock ($\rho_\mathcal{T}$, $\sigma_\mathcal{T}$ and τ).

- Very reasonable degree of price stickiness: average price durations of just over 1.5 quarters.

- habit formation parameter h and of the parameters of the policy rule are also broadly in line with other existing results

- small ι implying a very minor degree of inflation persistence.

2.5.4 Results: Model comparison

M1 vs M2 (stationary target, low indexation)

	AT1		linear	
truncation	**ML**	**emp. cov.**	**ML**	**emp. cov.**
0.1	1658.53	0.12	1631.21	0.16
0.2	1658.51	0.22	1631.22	0.28
0.3	1658.49	0.33	1631.21	0.39
0.4	1658.47	0.43	1631.21	0.48
0.5	1658.46	0.52	1631.22	0.57
0.6	1658.45	0.62	1631.24	0.65
0.7	1658.44	0.71	1631.25	0.73
0.8	1658.42	0.80	1631.28	0.80
0.9	1658.41	0.89	1631.31	0.88

3 Ongoing research

- DSGE for asset pricing: Amisano and Tristani (2008):
  - Use small DSGE to obtain SDF for assets (long term bonds)
  - Useful for obtaining inflation expectations and risk premia
  - Need a non linear model
  - Regime switching approach: variances of shocks jump across different states
- quality of the approximation: Amisano and Tristani (2007b)
- properties of different filtering techniques: Amisano and Tristani (2007c).

- issue of identification

- efficient MCMC: Amisano, Hoogerheide and van Dijk (2008, in progress)

- models with bounded rationality: learning