NASH-WALRAS EQUILIBRIA WITH PRIVATE INFORMATION ON BOTH SIDES

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Abstract. In competitive markets with asymmetric information individuals exchange assets and, simultaneously, choose actions that influence the assets' payoffs. Equilibria are known to exist if individuals can use their strategic power only on one side of the market. In this paper we propose an extension of the model in which individuals can be strategic both as buyers and as sellers, and we prove the existence of equilibria.

Key words: Nash, Walras, equilibrium, asymmetric information.

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Introduction

In the model of competitive equilibrium with incomplete markets, an asset is a promise contingent on the realization of a state of the world. It is assumed that the realized state of the world is fully observable, and its realization unaffected by the choices of individuals. For any given asset the sellers' obligations and the buyers' rights are thus fully specified and independent from their characteristics and choices, and the asset structure is exogenously given.

Dubey, Geanakoplos and Shubik (1990, 2006) generalized the model by allowing individuals to default on their promises. The default rate, and thus the actual payoff of the asset in a given state, depends on the endowment and utility of the seller (a problem of adverse selection), as well as on the seller's overall unobservable market actions (a problem of moral hazard).

The crucial observation in Dubey, Geanakoplos and Shubik (1990, 2006) is that large anonymous markets remain viable if every buyer expects to receive the *average* rate of return of each asset in each

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state, and anticipates it correctly. The assets' payoffs to the buyers become endogenous variables reflecting, at equilibrium, the sellers' private choices and characteristics [1].

Bisin and Gottardi (1998) and Minelli and Polemarchakis (1999) expand on this theme. In particular, Minelli and Polemarchakis (1999) allow the obligations of sellers to depend on their private characteristics, on their strategic choices and on the overall distribution of these choices in the economy. At a Nash-Walras equilibrium, an individual chooses her trades in asset and her strategy as a seller taking as given the assets' payoffs as a buyer, the prices, and the strategies of others. The assumption of uniform payoffs for the buyers is again sufficient for the existence of competitive equilibria.

Bisin Geanakoplos Gottardi Minelli and Polemarchakis (2001) prove that equilibria are generically constrained suboptimal: even if restricted to uniform taxes on the observable trades in assets, a planner can improve on market outcomes. This substantiates the claim in Greenwald and Stiglitz (1986) that constrained suboptimality is a pervasive property of equilibria with asymmetric information.

In contrast to standard models of incomplete markets, the asset structure at a Nash - Walras equilibrium reflects the distribution of information in the economy. Still, a crucial limitation is common to all quoted contributions: the one-sided treatment of private information and strategic behavior. Each seller may deliver differently on the same promise, but deliveries are pooled and every buyer of a given asset receives exactly the same (average) payoff in every state.

In many markets, both buyers and sellers have some private information, and they may try to exploit it. For example, workers are usually better informed on their skills, but firms may know more about working conditions and may have different abilities to select the best workers. Also, in the prototypical examples of the used cars market, some buyers may be better than others in avoiding 'lemons'.

Motivated by these examples, we extend the model by allowing both buyers and sellers to have different information and strategic abilities. A natural extension of the idea of uniform rationing allows for the existence of equilibria.

[1] A different, very influential model of general competitive equilibrium in which the possibility of default is taken into account has been proposed by Kehoe and Levine (1993). In their model default never occurs at equilibrium. The idea, following the approach of Prescott and Townsend (1984), is that no buyer will buy an asset from a seller who is likely to default. To enforce this type of equilibrium the buyer, or some central agency, must have complete information on the seller's trades.

The paper is organized as follows. In section 2 we introduce the notion of Nash-Walras equilibrium for our economy and we prove existence. A simple example is discussed in section 3 to illustrate the nature of the equilibrium.

1. The economy

Actions are $a \in \mathcal{A}$ where $\mathcal{A}$ is a non-empty, compact separable, metric space . Distributions of actions are $\nu \in \Delta(\mathcal{A})$. [2]

Commodities are $l = 1, \ldots, L$.

Trades in commodities are $z = (\ldots, z_l, \ldots) \in \mathbb{R}^L$.

Contracts for the delivery of commodities are $m = 1, \ldots, M$. Sales of contracts, portfolios of short positions, are $\phi = (\ldots, \phi_m, \ldots) \in \Phi = \{\phi : 0 \leq \phi \leq \overline{\phi}\}$, while purchases of contracts, portfolios of long positions, are $\theta = (\ldots, \theta_m, \ldots) \in \Theta = \{\theta : 0 \leq \theta\}$.

An individual is described by a continuous utility function, $u : \mathbb{R}^L \times \mathcal{A} \times \Delta(\mathcal{A}) \to \mathbb{R}$, a vector of initial endowments $w \in \mathbb{R}^L_{++}$, and continuous maps D and E, with domain $\mathcal{A} \times \Delta(\mathcal{A})$ and range $\mathcal{R}$, where $\mathcal{R}$ is a compact, convex subset of positive [3] matrices of dimension $L \times M$. The utility of the individual varies with (z, a, ν) : the net trade in commodities, the action of the individual, and the distribution of actions.

The matrix of deliveries on contracts sold by the individual is

$$D(a, \nu) = \{d_{l,m}\}^{l=1,\ldots,L}_{m=1,\ldots,M} \in \mathcal{R};$$

it varies with the delivery action of the individual and the distribution of delivery actions.

The a priori matrix of payoffs of contracts purchased by an individual, is

$$E(a, \nu) = \{e_{l,m}\}^{l=1,\ldots,L}_{m=1,\ldots,M} \in \mathcal{R};$$

it accordingly varies with the purchase action of the individual and the distribution of purchase actions.

The net trade in commodities by an individual is

$$z = K \boxtimes E\,(a, \nu)\,\theta - D(a, \nu)\phi,$$

[2]For $\mathcal{X}$, a non-empty, compact, separable metric space, $\Delta(\mathcal{X})$ denotes the set of Borel probability distributions on $\mathcal{X}$, which, when endowed with the topology of weak convergence, is itself a non-empty, compact, separable metrizable space. For n, a positive integer, Δ^n denotes the simplex of dimension $(n-1)$. When not stated otherwise, all the mathematical definitions and results can be found in Hildenbrand (1974).

[3]A positive matrix has all entries non - negative and at least one different from zero.

where K is a rationing martix of dimension $L \times M$, that is common to all the individuals, but depends on the equilibrium of the economy (see below). Hence individual i's "buying martix" $R^i\left(a^i, \nu\right) := K \boxtimes E^i\left(a^i, \nu\right)$ consists of a common part K and a private part $E^i\left(a^i \nu\right)$. The product $\boxtimes$ is defined entrywise, that is $r_{l,m} := k_{l,m} \cdot e_{l,m}$.

Prices of contracts are $q = (\ldots, q_m, \ldots)$, with domain Δ^M.

The budget set of an individual varies with (q, K, ν), the prices of contracts, the rationing martix and the distribution of actions; the budget set of an individual with characteristics (u, w, D, E) is

$$\beta(u, w, D, E, q, K, \nu) =$$
$$\{(a, \theta, \phi, z) \mid q(\theta - \phi) \leq 0,\ z = K \boxtimes E\,(a, \nu)\,\theta - D(a, \nu)\phi,\ z \leq -w\}$$

An individual chooses (a, θ, ϕ, z), an action, a sale of contracts, a purchase of contracts and a net trade in commodities, so as to

$$\max\ u(z, a, \nu), \quad \text{s.t. } (a, \theta, \phi, z) \in \beta(u, w, D, E, q, K, \nu).$$

The set of solutions to the optimization problem of an individual is $\psi(u, w, D, E, q, K, \nu)$.

The set of bounded, continuous utility functions of individuals is $\mathcal{U}$, $\mathcal{D}$ is the set of continuous functions of deliveries on contracts, $\mathcal{E}$ is the set of continuous function of choice options on contracts, all endowed with the topology of uniform convergence on compacta (compact-open topology, see Mas Colell (1985), p. 50). The set of characteristics of individuals is $\mathcal{U} \times \mathbb{R}^L_{++} \times \mathcal{D} \times \mathcal{E}$, a complete, separable, metric space.

Lemma. *Let $\mathcal{F}$ be a set of real valued continuous function defined on some separable metrizable space X. If a sequence $(f_n : n, \ldots) \subset \mathcal{F}$ converges to $f \in \mathcal{F}$ in the compact open topology, and if $C \subset X$ is a compact set and $(x_n : n, \ldots) \subset C$ converges to $x \in C$, then $f_n(x_n)$ converges to $f(x)$.*

An economy is $\mu \in \Delta(\mathcal{I})$, where $\mathcal{I} \subset \mathcal{U} \times \mathbb{R}^L_{++} \times \mathcal{D} \times \mathcal{E}$, the support of μ, is a compact subset of individuals with strictly monotone utilities: $z' > z \Rightarrow u(z', a, \nu) > u(z, a, \nu)$.

For individuals in $\mathcal{I}$, initial endowments lie in a compact set bounded from above by $\bar{w}$, hence we can restrict trades to the set $\mathcal{Z}$=$\{z : z \geq -\bar{w}\}$. The set of choices of individuals is then $\mathcal{C} = \mathcal{A} \times \Theta \times \Phi \times \mathcal{Z}$.

A joint distribution of characteristics and choices of individuals is $\tau \in \Delta(\mathcal{I} \times \mathcal{C})$.

For (q, K, τ), prices of contracts, payoffs of contracts purchased and a joint distribution of characteristics and choices of individuals, the best response set is[4]

$$\begin{aligned} B(q,K,\tau_{\mathcal{A}}) &= \{(u,w,D,E,a,\theta,\phi,z) \mid (a,\theta,\phi,z) \in \psi(u,w,D,E,q,K,\tau_{\mathcal{A}})\} \\ &\subset \mathcal{I} \times \mathcal{C} \end{aligned}$$

Given the way the rationing will be defined, to assure the existence of an equilibrium we have to make sure that no individual would receive infinite quantities of a commodity l if the were the first (the only one) who demands commoditiy l as a payoff from contract m. To insure this we impose the following condition. Define the set of couples (l,m) such that commodity l might be delivered by contract m:

$M^* := \{(l,m) \mid$ there is $S \subseteq I$, with $\mu(S) > 0$, such that for every $i \in S$ there is an action a and a distribution over actions, ν such that $D^i_{l,m}(a,\nu) > 0\}$.

We assume that for all commodities and contracts $(l,m) \in M^*$ there is a very small but positive $\eta_{l,m} > 0$ (with $\frac{1}{L} >> \eta_{l,m}$) such that almost every individual j, for each action, demands a at least $\eta_{l,m}$ units of commodity l as a payoff from one unit of contract m (that is for all actions a and all distributions over actions ν, $E^i_{l,m}(a,\nu) \geq \eta_{l,m}$, for allmost all $i \in I$).[5]

For all commodities $l \in L$ and contracts $m \in M$, we define a value that is an upper bound on the l,m-entry of the rationing matrix K:

$$\overline{k}_{l,m} := \frac{\max_{i \in I, a \in \mathcal{A}, \nu \in \Delta(\mathcal{A})} D^i_{l,m}(a,\nu)}{\eta_{l,m}},$$

if $(l,m) \in M^*$ and otherwise we set $\overline{k}_{l,m} := 0$.

Let $\mathcal{K} = \times_{l,m}[0,\overline{k}_{l,m}]$ be the space of possible rationing matrices.

Definition 1. *A Nash - Walras equilibrium for an economy, μ, is a joint distribution on the set of characteristics and the set of choices of individuals, τ^*, such that*

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[4]If σ is a distribution on a product set, $\ldots, \times\mathcal{B} \times \ldots$, then $\sigma_{\mathcal{B}}$ denotes the marginal distribution on $\mathcal{B}$.

[5]Note however that it is not enough that the condition imposed on the buyers side holds for some positive mass < 1 of buyers, because those buyers could just chose not to buy contract m. Then a positive mass of the other buyers could buy a positive amount of contract m without demanding commodity l, and some sellers could chose to deliver commodity l as a payoff for contract m sold. Then again the first deviator on the buyers side who would now chose to demand a positive amount of commodity l as a payoff from contract m would obtain an infinite amount of commodity l.

(1) *the marginal distribution of characteristics of individuals coincides with the distribution in the economy:*

$$\tau^*_{\mathcal{I}} = \mu;$$

(2) *there exist prices of contracts, q^*, and a rationing matrix of payoffs of contracts purchased, K^*, such that individuals optimize:*

$$\tau^*(B(q^*, K^*, \tau^*)) = 1;$$

(3) *the markets for contracts clear:*

$$\int_{\mathcal{I}\times\mathcal{C}} (\theta - \phi)\, d\tau^* = 0;$$

(4) *the rationing matrix of payoffs of contracts rations the deliveries on contracts:*

$$\int_{\mathcal{I}\times\mathcal{C}} e_{l,m}(a, \tau^*_{\mathcal{A}})\theta_m d\tau^* > 0 \quad \Rightarrow \quad k^*_{l,m} = \frac{\int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a, \tau^*_{\mathcal{A}})\phi_m d\tau^*}{\int_{\mathcal{I}\times\mathcal{C}} e_{l,m}(a, \tau^*_{\mathcal{A}})\theta_m d\tau^*},$$

(5) *the markets for commodities clear:*

$$\int_{\mathcal{I}\times\mathcal{C}} z d\tau^* = 0.$$

This is an extension of the notions of a competitive equilibrium for an economy and of a Nash equilibrium for a game to a large set of individuals.

Proposition 1. *Nash - Walras equilibria exist.*

Proof For $0 < \epsilon < 1/M$, $\Delta^M_\epsilon = \{q : \sum_{m=1}^M q_m = 1, q_m \geq \epsilon, m = 1, \dots M\}$.

The correspondence $\beta : \mathcal{I} \times \Delta^M_\epsilon \times \mathcal{K} \times \Delta(\mathcal{A}) \to \mathcal{C}$ is non - empty, compact valued and continuous. Non emptiness is clear. Let's argue that the image of every compact set is bounded. Indeed there is a bounded set which contains as subsets all the images of β. $\mathcal{A}$ and Φ are bounded sets. For every compact set of prices in Δ^M_ϵ, the θ must lie in the same compact set for every individual. The matrices E and D characterizing the individuals lie in a compact set, as do the initial endowments. Thus there exists a bounded set in which all z must lie.

The set $\beta(u, w, D, E, q, K, \nu)$ is then compact, because it is closed in the bounded set mentioned above. (A closed and bounded set of a separable complete metric space is compact).

To prove upper hemi-continuity, it's enough to show that the graph of β is closed. The only problem may be to show that if $E_n \to E$ and $(a_n, \nu_n) \to (a, \nu)$, then $E_n(a_n, \nu_n) \to E(a, \nu)$. To show this, fix $\delta > 0$. By continuity of E, there is n_1 such that for $n \geq n_1$ $\mid E(a_n, \nu_n) - E(a, \nu) \mid < \frac{\delta}{2}$; because $E_n \to E$ in the topology of uniform continuity, there is n_2 such that for $n \geq n_2 \mid E_n(a_n, \nu_n) - E(a_n, \nu_n) \mid < \frac{\delta}{2}$. Then, using the triangle inequality, $\mid E_n(a_n, \nu_n) - E(a, \nu) \mid < \delta$ for $n \geq \max[n_1, n_2]$. Analogously for D.

We now prove lower hemi-continuity. Take a sequence $((u_n, w_n, D_n, E_n, q_n, K_n, \nu_n) : n = 1, \ldots)$ converging to (u, w, D, E, q, K, ν), and a point $(a, \theta, \phi, z) \in \beta(u, w, D, E, q, K, \nu)$, such that $q(\theta - \phi) = 0$, and $z = K \boxtimes E\,(a, \nu)\,\theta - D(a, \nu)\phi$. There exists $\eta > 0$, such that [6] $a' = a$, $\theta' = 0$, $\phi' = \eta \mathbf{1}_M$, $z' = K \boxtimes E\,(a', \nu)\,\theta' - D(a', \nu)\phi' = -\eta D(a, \nu)\mathbf{1}_M$ satisfies $q(\theta' - \phi') = -\eta < 0$, and $-w < z'$. A sequence converging to $c = (a, \theta, \phi, z)$ is constructed by taking convex combinations. Since the action, a, remains fixed along the sequence, the possible non - convexity of the budget set does not interfere with the argument. Let

$$c_m = \left(a, \frac{1}{m}\theta' + (1 - \frac{1}{m})\theta, \frac{1}{m}\phi' + (1 - \frac{1}{m})\phi, \frac{1}{m}z' + (1 - \frac{1}{m})z\right)$$

For given m, there exists $n(m)$ such that, for all $n \geq n(m)$, $c_m \in \beta(u_n, w_n, D_n, E_n, q_n, K_n, \nu_n)$. We construct the converging sequence by taking $c_n = c_m$ for all $n(m) \leq n < n(m+1)$, and c_n any point in $\beta(u_n, w_n, D_n, E_n, q_n, K_n, \nu_n)$ if $n < n(1)$.

The set $\mathcal{C}_\epsilon = \bigcup_{(i,q,K,\nu) \in \mathcal{I} \times \Delta^M_\epsilon \times \mathcal{K} \times \Delta(\mathcal{A})} \beta(i, q, K, \nu)$ is compact. Since the domain of β is compact and β is u.h.c., the graph of β is a compact set. $\mathcal{C}_\epsilon$ is the projection of the graph on $\mathcal{C}$ and hence compact.

The correspondence $B : \Delta^M_\epsilon \times \mathcal{K} \times \Delta(\mathcal{A}) \to \mathcal{I} \times \mathcal{C}_\epsilon$ is non - empty, upper hemi - continuous and compact valued.

It is non-empty because each individual is maximizing a continuous function on non-empty compact set.

We now show that B is upper hemi-continuous. Because the range of B is compact, we just need to prove that it has a closed graph. Take a sequence, $((q_n, K_n, \tau_n) : n = 1, \ldots)$, converging to $(q, K, \tau_{\mathcal{A}})$, and a sequence $((u_n, w_n, D_n, E_n, a_n, \theta_n, \phi_n, z_n) : n = 1, \ldots)$ converging to $(u, w, D, E, a, \theta, \phi, z)$ with $(a_n, \theta_n, \phi_n, z_n) \in \psi(u_n, w_n, D_n, E_n, q_n, K_n, \tau_{\mathcal{A},n})$, for $n = 1, \ldots$. Because β is u.h.c., $(a, \theta, \phi, z) \in \beta(u, w, D, E, q, K, \tau_{\mathcal{A}})$. If $(a, \theta, \phi, z) \notin \psi(u, w, D, E, q, K, \tau_{\mathcal{A}})$, by the l.h.c. of β, there exists a sequence $(a'_n, \theta'_n, \phi'_n, z'_n) \in \beta(u_n, w_n, D_n, E_n, q_n, K_n, \tau_{\mathcal{A},n})$ that converges to a point (a', θ', ϕ', z'), with $u(z', a', \tau_{\mathcal{A}}) > u(z, a, \tau_{\mathcal{A}})$. Since $u_n(z_n, a_n, \tau_{\mathcal{A},n}) \geq u_n(z'_n, a'_n, \tau_{\mathcal{A},n})$, this contradicts the convergence of the sequence $(u_n : 1, \ldots)$ to u in the compact open topology.

The set $\mathcal{T}_\epsilon \subset \Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$, such that, if $\tau \in \mathcal{T}_\epsilon$, then $\tau_{\mathcal{I}} = \mu$, is obviously convex and it is compact because it is a closed subset (Hildenbrand (1974), (27) on p. 48 : if a sequence of measures converges weakly, then so do the marginals) of a compact set.

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[6] "$\mathbf{1}_K$"denotes the column vector of 1's of dimension K.

The correspondence $\Phi_{1,\epsilon} : \Delta_\epsilon^M \times \mathcal{K} \times \mathcal{T}_\epsilon \to \mathcal{T}_\epsilon$ defined by

$$\Phi_{1,\epsilon}(q, K, \tau) = \{\tau' \in \mathcal{T}_\epsilon : \tau'(B(q, K, \tau_{\mathcal{A}})) = 1\}$$

is non - empty, convex compact valued valued and upper hemi-continuous.

Convex valuedness is clear.

Let us prove non emptiness. For any $(q, K, \tau) \in \Delta_\epsilon^M \times \mathcal{K} \times \mathcal{T}_\epsilon$, $\Phi_{1,\epsilon}(q, K, \tau) \neq \emptyset$. For any (q, K, τ), the correspondence $\psi(\,\cdot\,, q, K, \tau_{\mathcal{A}}) : \mathcal{I} \to \mathcal{C}_\epsilon$ is non-empty valued and upper hemi-continuous. Therefore it admits a measurable selection $s : \mathcal{I} \to \mathcal{C}_\epsilon$ (Aliprantis-Border (1999), Theorem 17.13, p. 567). We construct a measure $\tau_s \in \Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$ such that the marginal of τ_s on $\mathcal{I}$ is μ. To do this, define, for any measurable rectangle $A \times B \subset \mathcal{I} \times \mathcal{C}_\epsilon$, $\tau_s(A \times B) := \mu(A \cap s^{-1}(B))$. This determines uniquely a σ-additive probability measure in $\Delta(\mathcal{I} \times \mathcal{C}_\epsilon)$ which obviously has the property that the marginal on $\mathcal{I}$ is μ.

We now prove that $\Phi_{1,\epsilon}$ is upper hemi-continuous. Because the range of $\Phi_{1,\epsilon}$ is compact, we just need to prove that it has a closed graph. If a sequence, $((q_n, K_n, \tau_n) : n = 1, \ldots)$, converges to (q, K, τ), and a sequence, $(\tau'_n : n = 1, \ldots)$, such that $\tau'_n \in \Phi_{1,\epsilon}(q_n, K_n, \tau_n)$, converges to τ', then $\tau' \in \Phi_{1,\epsilon}(q, K, \tau)$. If not, $\tau'(B(q, K, \tau)) < 1$. $B(q, K, \tau)$ is closed in the metrizable space $\mathcal{I} \times \mathcal{C}_\epsilon$, therefore there exist open sets U and V such that $B(q, K, \tau) \subset V \subset \overline{V} \subset U$ and $\tau'(U) < 1$. Since the correspondence B is upper hemi - continuous, there exists $\overline{n}$, such that $B(q_n, K_n, \tau_n) \subset V$, , for $n = \overline{n}, \ldots$. Since $\tau'_n(B(q_n, K_n, \tau_n)) = 1$, $\tau'_n(\overline{V}) = 1$ for $n = \overline{n}, \ldots$. Since the sequence $(\tau'_n : n = 1, \ldots)$ converges weakly to τ, $\limsup_n \tau'_n(\overline{V}) \leq \tau'(\overline{V})$ (Hildenbrand (1974), iii) of (26), p.48). Then $\tau'(\overline{V}) = 1$ and $\tau'(U) = 1$, a contradiction.

The function $\Phi_{2,\epsilon} : \mathcal{T}_\epsilon \to \mathcal{K}$ is defined by

$$\Phi_{2,\epsilon,l,m}(\tau) := \min \left\{ \overline{k}_{l,m}, \frac{\epsilon \overline{k}_{l,m} + \int_{\mathcal{I} \times \mathcal{C}_\epsilon} d_{l,m}(a, \tau_{\mathcal{A}}) \phi_m d\tau}{\epsilon + \int_{\mathcal{I} \times \mathcal{C}_\epsilon} e_{l,m}(a, \tau_{\mathcal{A}}) \theta_m d\tau} \right\},$$

where $\overline{K} \in \mathcal{K}$ is the matrix of payoffs of contracts introduced before Definition 1. Given that $\epsilon > 0$, to prove continuity it is enough to show that the two integrals are continuous functions of the distribution τ. Consider $\int_{\mathcal{I} \times \mathcal{C}} d_{l,m}(a, \tau_{\mathcal{A}}) \phi_m d\tau$ (the same argument holds for $\int_{\mathcal{I} \times \mathcal{C}} e_{l,m}(a, \tau_{\mathcal{A}}) \theta_m d\tau$). Take a sequence $\tau_n (n = 1, \ldots) \in \mathcal{T}_\epsilon$ converging to $\tau \in \mathcal{T}_\epsilon$. We want to show that, fixing any $\delta > 0$, there is $\overline{n}$ such that for all $n \geq \overline{n}$

$$\left| \int_{\mathcal{I} \times \mathcal{C}} d_{l,m}(a, \tau_{\mathcal{A}}) \phi_m d\tau - \int_{\mathcal{I} \times \mathcal{C}} d_{l,m}(a, \tau_{\mathcal{A},n}) \phi_m d\tau_n \right| < \delta$$

The previous expression can be written as

$$| \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m d\tau - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m d\tau_n +$$

$$\int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m d\tau_n - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m d\tau_n |< \delta$$

Because $d_{l,m}(a,\tau_{\mathcal{A}})\phi_m$ is a bounded continuous function of the integrating variables (i,c) on $\mathcal{I}\times\mathcal{C}_\epsilon$ (the restriction to $\mathcal{C}_\epsilon$ is important to bound θ when considering $\int_{\mathcal{I}\times\mathcal{C}} e_{l,m}(a,\tau_{\mathcal{A}})\theta_m d\tau$), and τ_n converges to τ in the weak topology, there exists n_1 such that, for $n \geq n_1$,

$$| \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m d\tau - \int_{\mathcal{I}\times\mathcal{C}} d_{l,m}(a,\tau_{\mathcal{A}})\phi_m d\tau_n |< \frac{\delta}{2}.$$

As a function from the compact set $\mathcal{I}\times\mathcal{C}_\epsilon\times\mathcal{T}_\epsilon$ to R^L_+, the expression $d_{l,m}(a,\tau_{\mathcal{A}})\phi_m$ is uniformly continuous. Then there exists n_2 such that, for $n \geq n_2$,

$$| d_{l,m}(a,\tau_{\mathcal{A}})\phi_m - d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m |< \frac{\delta}{2}$$

for all (i,c).

Then for any $n \geq n_2$

$$\int | d_{l,m}(a,\tau_{\mathcal{A}})\phi_m - d_{l,m}(a,\tau_{\mathcal{A},n})\phi_m |< \frac{\delta}{2}\int d\tau_n = \frac{\delta}{2}$$

Let then $\overline{n} = \max[n_1, n_2]$.

Define the correspondence

$$\Phi_{3,\epsilon} : \mathcal{T}_\epsilon \to \Delta^M_\epsilon$$

by $\Phi_{3,\epsilon}(\tau) = \arg\max_{\Delta^M_\epsilon} q \int_{\mathcal{I}\times\mathcal{C}_\epsilon}(\theta-\phi)d\tau$. Clearly, it is non-empty and convex valued. To see that it is upper hemi-continuous, first notice that, as a function of τ, $\int_{\mathcal{I}\times\mathcal{C}_\epsilon}(\theta-\phi)d\tau$ is continuous (weak convergence) and it has a compact range in $\mathbb{R}^M$. Then, by standard argument, the graph of $\Phi_{3,\epsilon}$ is closed.

The correspondence $\Phi_\epsilon = \Phi_{1,\epsilon}\times\Phi_{2,\epsilon}\times\Phi_{3,\epsilon} : \Delta^M_\epsilon\times\mathcal{K}\times\mathcal{T}_\epsilon \to \Delta^M_\epsilon\times\mathcal{K}\times\mathcal{T}_\epsilon$ is non - empty, convex, compact valued and upper hemi - continuous; therefore, it has a fixed point, $(q_\epsilon, K_\epsilon, \tau_\epsilon)$.

For $\epsilon = 1/n$, the sequence of fixed points is $((q_n, K_n, \tau_n) : n = M+1, \ldots,)$.

Aggregating the budget constraints and using the definition of $\Phi_{3,\epsilon}$, $q\int_{\mathcal{I}\times\mathcal{C}}(\theta-\phi)d\tau_n \leq q_n\int_{\mathcal{I}\times\mathcal{C}}(\theta-\phi)d\tau_n \leq 0$, for all $q \in \Delta^M_{1/n}$. Take $q = \mathbf{1}_M(1/M)$, then $\sum_{m=1}^M \int_{\mathcal{I}\times\mathcal{C}}\theta_m d\tau_n \leq \sum_{m=1}^M \int_{\mathcal{I}\times\mathcal{C}}\phi_m d\tau_n \leq \sum_{m=1}^M \overline{\phi}_m = \overline{\theta}$. Let $\mathcal{C}_{\overline{\theta}}$ be the compact subset of $\mathcal{C}$ where we impose the restriction that, for all m, $\theta_m \leq \overline{\theta}$ (remember that $z = K \boxtimes E\,(a, \tau_{\mathcal{A}})\,\theta - D(a, \tau_{\mathcal{A}})\phi$ and that E, D and ϕ are bounded, so this provides also a bound on z).

We want to show that the sequence $\tau_n(n = M+1, \ldots)$ is a tight family of measures (see Hildenbrand (1974, p.49)). If not, there exists $\delta > 0$ such that, for any compact $F \subset \mathcal{I} \times \mathcal{C}$, there exists n with $\tau_n(F) \leq 1 - \delta$. Fix $\delta > 0$ and consider $F = \mathcal{I} \times \mathcal{C}_{\hat{\theta}}$ where $\hat{\theta} = \frac{2\overline{\theta}M}{\delta}$. Then $\tau_n(\mathcal{I} \times \mathcal{C} \setminus \mathcal{C}_{\hat{\theta}}) > \delta$. Given that $\theta \geq 0$, this would imply that, for some m, $\int_{\mathcal{I}\times\mathcal{C}} \theta_m d\tau_n > \overline{\theta}$, a contradiction.

Using the fact that Δ^M and $\mathcal{K}$ are compact, the property of tight family of measures stated in Hildenbrand (1974) (31) on p. 49, and the fact that $\Delta(\mathcal{I} \times \mathcal{C})$ is complete, we can extract from the sequence of fixed points $((q_n, K_n, \tau_n) : n = M+1, \ldots)$ a subsequence, which we denote again $((q_n, K_n, \tau_n) : n = M+1, \ldots)$, converging to a point $(q^*, K^*, \tau^*) \in \Delta^M \times \mathcal{K} \times \Delta(\mathcal{I} \times \mathcal{C})$.

We want to show that the limit point (q^*, K^*, τ^*) is an equilibrium.

Let us first argue that markets for contracts clear. Given that $0 \notin \mathcal{R}$, M^* is non empty and, at each fixed point there exists at least one (l, m) with $\Phi_{2,\epsilon,l,m}(\tau_\epsilon) > 0$ and for almost all $i \in \mathcal{I}$ and every $a \in \mathcal{A}$, $E^i_{l,m}(a, \tau_{\epsilon,\mathcal{A}}) > 0$. E is a positive matrix and the utility function strictly monotone in consumption, so that at each fixed point almost all individuals are spending all their revenue from the sales of contracts. Taking limits, $q^* \int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)d\tau^* = 0$, and, using the definition of $\Phi_{3,\epsilon}$, $q \int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)d\tau^* \leq 0$, for all $q \in \cup_\epsilon \Delta^M_\epsilon = \Delta^M_{++}$. This implies that $\int_{\mathcal{I}\times\mathcal{C}}(\theta - \phi)d\tau^* \leq 0$. If, for some m, $\int_{\mathcal{I}\times\mathcal{C}}(\theta_m - \phi_m)d\tau^* < 0$, then $q^*_m = 0$. Consider the following modification $\hat{\tau}^*$ of τ^*. Write $\mathcal{C}$ as $\mathcal{C} = \mathcal{A} \times \Theta^{-m} \times \mathbb{R}_+ \times \Phi \times \mathcal{Z}$. Let the per capita excess supply of asset m be $\hat{\theta} = -\int_{\mathcal{I}\times\mathcal{C}}(\theta_m - \phi_m)d\tau^*$. For any measurable rectangle $A \times B \in \mathcal{I} \times \mathcal{C}$ with $A \subset \mathcal{A} \times \Theta^{-m} \times \Phi \times \mathcal{Z}$ and $B \subset \mathbb{R}_+$, let $\hat{\tau}^*(A \times B) = \tau(A \times (B - \hat{\theta}) \cap \mathbb{R}_+)$. This modification assures market clearing for contracts without changing prices nor the other choices of the individuals.

Let us now show that, at the limit, there is also market clearing for commodities. From the budget constraints of individuals and the definition of $\Phi_{2,\epsilon}$ we have, for any $l \in L$

$$\int_{\mathcal{I}\times\mathcal{C}} z_l d\tau_\epsilon = \sum\nolimits_m [\Phi_{2,\epsilon,l,m}(\tau_\epsilon) \int_{\mathcal{I}\times\mathcal{C}} e_{lm}(a, \tau_{\mathcal{A},\epsilon})\theta_m d\tau_\epsilon - \int_{\mathcal{I}\times\mathcal{C}} d_{lm}(a, \tau_{\mathcal{A},\epsilon})\phi_m d\tau_\epsilon]$$

To simplify notation, let $\int_{\mathcal{I}\times\mathcal{C}} e_{lm}(a, \tau_{\mathcal{A},\epsilon})\theta_m d\tau_\epsilon = D^\epsilon_{lm}$ and $\int_{\mathcal{I}\times\mathcal{C}} d_{lm}(a, \tau_{\mathcal{A},\epsilon})\phi_m d\tau_\epsilon = S^\epsilon_{lm}$, both non negative numbers.

We claim that, for any $\epsilon > 0$, at a fixed point $\overline{k}_{lm} D^\epsilon_{lm} - S^\epsilon_{lm} \geq 0$. If $S^\epsilon_{lm} = 0$ this is clear. If $S^\epsilon_{lm} > 0$, it must be that $(lm) \in M^*$ and

$\int \phi_m(\epsilon) > 0$. By market clearing in contracts, $\int \theta_m(\epsilon) = \int \phi_m(\epsilon) > 0$, and because $(lm) \in M^*$, $\int_{\mathcal{I}\times\mathcal{C}} e_{lm}(a, \tau_{\mathcal{A},\epsilon})\theta_m d\tau_\epsilon > 0$. Thus, if $\overline{k}_{lm} D^\epsilon_{lm} < S^\epsilon_{lm}$we could divide both sides by D^ϵ_{lm} and obtain

$$\overline{k}_{lm} < \frac{S^\epsilon_{lm}}{D^\epsilon_{lm}} \leq \frac{\max_{i\in I, a\in\mathcal{A}, \nu\in\Delta(\mathcal{A})} D^i_{l,m}(a,\nu) \int \phi_m(\epsilon)}{\eta_{l,m} \int \theta_m(\epsilon)} = \overline{k}_{lm}$$

a contradiction.

Because $\overline{k}_{lm} D^\epsilon_{lm} - S^\epsilon_{lm} \geq 0$, for every $\epsilon > 0$, at a fixed point:

$$\Phi_{2,\epsilon,l,m}(\tau_\epsilon) = \frac{\epsilon \overline{k}_{lm} + S^\epsilon_{lm}}{\epsilon + D^\epsilon_{lm}}.$$

Thus

$$\begin{array}{rcl} 0 & \leq & \int_{\mathcal{I}\times\mathcal{C}} z_l d\tau_\epsilon \\ & = & \sum_m [\frac{\epsilon\overline{k}_{lm} + S^\epsilon_{lm}}{\epsilon + D^\epsilon_{lm}} D^\epsilon_{lm} - S^\epsilon_{lm}] \\ & = & \sum_m [\frac{\epsilon}{\epsilon + D^\epsilon_{lm}} (\overline{k}_{lm} D^\epsilon_{lm} - S^\epsilon_{lm})] \\ & \leq & \sum_m [\frac{\epsilon D^\epsilon_{lm}}{\epsilon + D^\epsilon_{lm}} \overline{k}_{lm}] \\ & \leq & \epsilon \sum_m \overline{k}_{lm} \end{array}$$

and, at the limit, $\int_{\mathcal{I}\times\mathcal{C}} z d\tau^* = 0$.

It remains to show that, at the limit, individuals are indeed optimizing, $\tau^*(B(q^*, K^*, \tau^*)) = 1$.

To simplify notation, let us define $B_n = B(q_n, K_n, \tau_n)$ and $B^* = B(q^*, K^*, \tau^*)$.

The proof proceeds in two steps. In the first one we fix $\epsilon > 0$ and we show that there exists a compact set $\overline{V}$ such that, if we define $\overline{B}_n = B_n \cap \overline{V}$, there is a subsequence converging to some $\overline{B}$ such that $\tau^*(\overline{B}) > 1 - \epsilon$.

In the second step we show that $\overline{B} \subset B^*$.

Since $\mathcal{I}$ and $\mathcal{C}$ are separable and complete metric spaces, for any given $\delta > 0$ there exist open sets V and U, with $\overline{U}$ compact such that $V \subset \overline{V} \subset U \subset \overline{U}$ and $\tau^*(V) > 1 - \delta$. Becuase τ_n converges (weakly) to τ^*, this implies $\text{liminf}_n \tau_n(V) \geq \tau^*(V) > 1 - \delta$. That is, there exists $\overline{n}$ such that, for all $n \geq \overline{n}$ $\tau_n(V) > 1 - \delta$. All along the sequence of fixed points we have $\tau_n(B_n) = 1$, so it must be that, for $n \geq \overline{n}$, $\tau_n(\overline{B}_n) = \tau_n(B_n \cap \overline{V}) > 1 - \delta$. Since $\overline{U}$ is a compact metric space, its non empty compact subsets, endowed with the Hausdorff metric, form a compact metric space. Because each B_n is non empty and closed, the $\overline{B}_n$'s are nonempty compact subsets of $\overline{U}$. We can therefore find a subsequence $(\overline{B}_m : m = 1, \ldots)$ converging to some compact $\overline{B} \subset \overline{U}$.

For given $\epsilon > 0$, choose $\delta < \epsilon$ and suppose that $\tau^*(\overline{B}) < 1-\epsilon < 1-\delta$. Since $\overline{B}$ is compact, there exists an open set W such that $\overline{B} \subset W$ and

$\tau^*(\overline{W}) < 1-\epsilon$. Then we also have $\overline{B} \subset (W \cap U)$ and $\tau^*(\overline{W} \cap \overline{U}) < 1-\epsilon$. Because $\overline{B}_m$ converges to $\overline{B}$, there exists $\overline{m}$ such that, for all $m \geq \overline{m}$, $\overline{B}_m \subset (W \cap U)$. But then, for all $m \geq \overline{m}$, $1-\delta < \tau_m(\overline{B}_m) \leq \tau_m(\overline{W} \cap \overline{U})$. Using again the weak convergence of τ_m to τ^*, we conclude $1-\delta < \text{limsup}_m \tau_m(\overline{W} \cap \overline{U}) \leq \tau^*(\overline{W} \cap \overline{U}) < 1-\epsilon$, a contradiction.

Let us now show that $\overline{B} \subset B^*$. Because $(\overline{B}_m : m = 1, \ldots)$ converges to $\overline{B}$ in the Hausdorff metric, for any given point $(i, c) \in \overline{B}$ we can construct a sequence (i_m, c_m) converging to (i, c) with $(i_m, c_m) \in B_m$. We need to show that $(i, c) \in B^*$. That is, we have to show the closed graph property of the correspondence $B : \Delta^M \times \mathcal{K} \times \Delta(\mathcal{A}) \to \mathcal{I} \times \mathcal{C}$. We already proved the closed graph property on the restricted domain $\Delta_\epsilon^M \times \mathcal{K} \times \Delta(\mathcal{A})$. It is then enough to observe that the same argument works here, because the budget correspondence β has a closed graph and is lower hemi-continuous even on the extended domain $\Delta^M \times \mathcal{K} \times \Delta(\mathcal{A})$. □

2. An example

$I = \{1, 2, 3\}$, $L = 3$, $M = 2$, $\mathcal{A} = \mathcal{B} = [0, 1]$.

Finitely many individuals. We will simplify by restricting the individual's buying and selling matrices to depend *only* on her choice.

Endowments:

$w^1 = (12, 0, 0)$, $w^2 = (0, 12, 0)$, $w^1 = (0, 0, 12)$,

Utility of $i = 1, 3$:

$u = \ln x_0 + \ln x_b + 2 \ln x_g$

Utility of $i = 2$:

$u = \ln x_0 + 2 \ln x_g$

Actions and Payoffs

$$E^1 = D^1 = \begin{matrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix}$$

and for $i = 2, 3$:

$$E^i(b^i) = \begin{matrix} 1 & 0 \\ 0 & b^i \\ 0 & 1-b^i \end{matrix}$$

$$D^i(a^i) = \begin{matrix} 1 & 0 \\ 0 & a^i \\ 0 & 1-a^i \end{matrix}$$

Rationing matrix:

$$K = \begin{matrix} 1 & 0 \\ 0 & \beta \\ 0 & \gamma \end{matrix}$$

Interpretation:

x_0 is 'money', x_b and x_g are two qualities of cherries, that Mr. 1 cannot distinguish. Mr. 2 and 3 are able to distinguish the two qualities of cherries, and can act strategically both as buyers and as sellers.

Budgets

Assets:

$\theta_1^i + q\theta_2^i \leq \phi_1^i + q\phi_2^i$

Goods:

Mr.1:
$x_0^1 = 12 - \phi_1^1 + \theta_1^1$
$x_b^1 = \frac{1}{2}\beta\theta_2^1 - \frac{1}{2}\phi_2^1$
$x_g^1 = \frac{1}{2}\gamma\theta_2^1 + \frac{1}{2}\phi_2^1$

Mr.2:
$x_0^2 = \theta_1^2 - \phi_1^2$
$x_b^2 = 12 + b^2\beta\theta_2^2 - a^2\phi_2^2$
$x_g^2 = (1 - b^2)\gamma\theta_2^2 + (1 - a^2)\phi_2^2$

Mr.3:
$x_0^3 = \theta_1^3 - \phi_1^3$
$x_b^3 = b^3\beta\theta_2^3 - a^3\phi_2^3$
$x_g^3 = 12 + (1 - b^3)\gamma\theta_2^3 + (1 - a^3)\phi_2^3$

Bounds on sales:

$D^i\phi^i \leq w^i$

Thus:

$a^2 = 1$, $a^3 = 0$.

Mr. 2 has no utility for x_1, thus $b^2 = 0$.

We are left to find asset trades (θ^i, ϕ^i), prices $q = (1, q)$, action b^3 and rationing coefficients β and γ.

Solution:

Mr.1:
$\theta_1^1 = 0, \phi_1^1 = 9$
$\theta_2^1 = \frac{9}{q}, \phi_2^1 = 0$

Mr.2:
$\theta_1^2 = 4q, \phi_1^2 = 0$
$\theta_2^2 = 8, \phi_2^2 = 12$

Mr.3:
$\theta_1^3 = 3q, \phi_1^3 = 0$
$\theta_2^3 = 3, \phi_2^3 = 6$

Equilibrium prices and rationing:

$\hat{q} = \frac{9}{7}$, $\hat{\beta} = \frac{24}{13}$, $\hat{\gamma} = \frac{12}{23}$

Action:
$b^3 = 1$

Mr. 1 buys $\hat{\theta}_2^1 = 7$ units of the cherry-asset and ends up with

$x_b^1 = (\frac{1}{2}\hat{\beta})\ 7 = 6.8$ units of b-cherries
$x_g^1 = (\frac{1}{2}\hat{\gamma})\ 7 = 1.8$ units of g-cherries

Mr. 2 and 3 sell the cherry-asset, and buy it back while doing 'cherry-picking'. Mr. 2 only picks g-cherries, Mr. 3 only picks b-cherries:

$\theta_2^2 = 8$, $\phi_2^2 = 12$
$x_b^2 = (0\hat{\beta})\ 7 = 0$
$x_g^2 = (1\hat{\gamma})\ 8 = 4.2$

$\theta_2^3 = 3$, $\phi_2^3 = 6$
$x_b^3 = (1\hat{\beta})\ 3 = 5.2$
$x_g^3 = (0\hat{\gamma})\ 3 = 0$

The cherry-asset sells on an anonymous market for a price $\hat{q} = \frac{9}{7}$, but different individuals face different rights and obligations, depending on their strategic power. On the buying side:

$$R^1 = K \boxtimes E^1 = \begin{matrix} 1 & 0 \\ 0 & 0.9 \\ 0 & 0.26 \end{matrix}$$

$$R^2 = K \boxtimes E^2 = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 0.52 \end{matrix}$$

$$R^3 = K \boxtimes E^3 = \begin{matrix} 1 & 0 \\ 0 & 1.8 \\ 0 & 0 \end{matrix}$$

It is as if the cherry-asset came itself in two qualities, with endogenous payoffs $\begin{matrix} 0 & 0 \\ \hat{\beta} & 0 \\ 0 & \hat{\gamma} \end{matrix}$

Mr.1 receives a mix of the two assets, while Mr. 2 and Mr. 3 can 'cherry-pick' the asset.

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